Preempting Preemptive Investment*

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This Draft: November, 2009

Abstract

This paper derives subgame-perfect equilibria for a dynamic, infinite horizon capital accumulation game in which investment is irreversible and demand is stochastic. I pay special attention to a class of strategies characterized by the small firm preempting, as cheaply as possible, the larger firm's preemptive investment behavior. The results have strong implications for the Stackelberg-like results found in Stackelberg-Spence-Dixit models. In particular, the intuition that a "leader" can profitably foreclose the market through preemptive investment depends on extreme, unrealistic "static market" assumptions of no demand growth, no demand uncertainty, and non-depreciating capital.

Keywords: capital accumulation, preemption, market structure, Stackelberg, dynamic games, real options.

JEL Classification: L13, D43, G12.

*I would like to thank Peter DeMarzo, Rob Gertner, Steve Grenadier, Milena Novy-Marx, Morten Sorenson and Ioanid Rosu for discussions and comments. Financial support from the Center for the Research in Securities Prices at the University of Chicago Graduate School of Business is gratefully acknowledged.

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1 Introduction

This paper shows that the Stackelberg-like results found in Stackelberg-Spence-Dixit (SSD) models (Spence (1977, 1979), Dixit (1979, 1980)), in which a "leader" can preemptively invest in order to foreclose the "follower's" investment opportunities, depend crucially on "static" market assumptions made in these models. In dynamic markets, in which demand grows or capital depreciates, firms' concerns extend beyond current investment opportunities. Firms continually compete over future opportunities, and as a result the leader cannot profitably discourage investment through preemptive behavior.

Canonical SSD models make static market assumptions for the sake of tractability. Consumers with mature demands are served by firms in their infancy: the market springs into existence with a demand curve that is fixed in perpetuity, but firms are endowed with little capital. "Growth" in these models references the capital employed in production. Because demand never increases and capital never depreciates, at some point firms stop investing and the market stops growing. Strategically the game ends when both firms' capacities meet or exceed their Cournot reaction curves, from which point on all state variables are fixed in perpetuity. Firms thus race to a "Stackelberg point," with the leader investing beyond the Cournot point, which diminishes the profitability of, and consequently reduces, the follower's investment. This forces the follower to invest quickly, in order to limit the extent to which the leader can foreclose the market through the price externality channel.

In dynamic settings, however, there is no Stackelberg point to which firms can race. Firms are always competing over future investment opportunities, and this fundamentally alters the nature of a firm's dynamic programming problem. The result is equilibria that differ dramatically from those in the "static market" game, much as equilibria in the infinitely repeated prisoner's dilemma differ from those in the finitely repeated game. When demand grows or capital depreciates, competition over future investment opportunities forces a follower to invest past its Cournot reaction curve, to prevent the "leader" from preempting *future* investment. The small firm effectively follows a strategy of preempting the large firm's preemptive investment behavior, making Stackelberg play unprofitable to the leader. Ironically, it is precisely the commitment power of capital, which we generally associate with Stackelberg behavior, that drives this result. In a static setting, the commitment power of capital allows the leader to credibly preempt the follower's investment; in a dynamic setting, it allows the follower to credibly preempt the leader's preemptive investment.

This paper illustrates these results in an oligopoly version of the dynamic, stochastic capital accumulation model of Leahy (1993). Conceptually this game represents a stochastic demand version of those studied by Spence (1979) and Fudenberg and Tirole (1983). Firms compete over the long-run by accumulating capacity, facing goods prices determined by short-run product-market competition. Grenadier (2002) exhibits a symmetric Cournot-Nash equilibrium of this game, and finds that competition drastically erodes the value of the option to delay investment, leading to investment near the zero net present value threshold. This conclusion has been broadly influential in the real-options literature. Extensions of the game's framework provide a useful laboratory for analyzing risk/return dynamics in equilibrium (Aguerrevere (2006), Novy-Marx (2007)).

Back and Paulson (2009) offers a critique of the equilibrium developed in Grenadier (2002), noting that it is "open-loop," *i.e.,* that it employs precommitment strategies, and argues that preemption concerns render this equilibrium impracticable.[1] They show that perfectly competitive outcomes can be produced by a dynamically consistent strategy.

I derive Markov perfect equilibria of the game, paying particular attention to a class of strategies characterized by the small firm preempting, as cheaply as possible, the larger firm's preemptive investment. The class includes a member that produces perfectly competitive investment behavior, like that found in Back and Paulson (2009). The Pareto-dominant

[1] An open-loop equilibrium is one in which players simultaneously precommit to their entire path of play at the start of the game. These equilibria are really static, in the sense that players make decisions at only one point in time. Because players employing open-loop strategies cannot alter their behavior in response to off-equilibrium play by their opponents in the course of the game, even if it would be optimal for them to do so, these equilibria raise concerns regarding dynamic consistency (*i.e.,* sub-game perfection). A "closed-loop" (or "feedback") equilibrium is a Nash equilibrium in state-dependent strategies.

Note that Spence (1979) also employs open-loop strategies (see Reinganum and Stokey (1985)).

member of this class, however, yields Cournot outcomes. In fact, a second message of the paper, similar in spirit to that of Kreps and Scheinkman (1983), concerns the robust nature of Cournot behavior. While the Cournot *strategy,* under which a firm only accounts for the price externality associated directly with its own investment, does not itself support a closed-loop equilibrium, a particularly natural Markov perfect equilibrium supports Cournot *behavior.*

While a Markov perfect equilibrium yields Cournot outcomes, and a reasonable selection criterion (Pareto-dominance) argues in favor of this particular outcome if firms play "preempting preemption" strategies, I am agnostic on the issue of equilibrium selection more broadly. Even with the Markov restriction, the set of possible equilibria extends beyond this class. For example, a "collusive" strategy, which I also examine in this paper, supports shared monopoly outcomes. This collusive behavior relies, however, on a Markov punishment mechanism that is particularly "active," in the sense that it calls for an instantaneous tit-for-tat response to any deviation from equilibrium play. This mechanism may be unattractive, however, because it depends critically on the perfect information nature of the game.

Finally, note that I am *not* disputing the existence of a first mover advantage. In some industries firms undoubtedly have an incentive to invest early and aggressively. Positive network externalities in product markets provide an incentive for a leader to establish market share. This paper shows, however, that a firm cannot generally permanently capture market share by foreclosing the market through the price externality channel, except under narrow, unrealistic static market assumptions.

The remainder of the paper is organized as follows. Section 2 presents the game. Section 3 discusses equilibria that have previously appeared in the literature, the competitive equilibrium of Leahy (1993) and the "Cournot" equilibrium of Grenadier (2002). Section 4 derives non-trivial closed-loop equilibria to the game. Section 5 discusses implications for Stackelberg behavior. Section 6 concludes.

2 Game

The game is a two player capital accumulation game, in which product market competition determines prices in the short-run and firms compete in the long-run by investing in new capacity.[2] The players, risk-neutral firms, employ capital to produce a homogeneous good. Consumers have downward sloping (iso-elastic) demand for firms' output. The level of demand varies stochastically over time. Firms increase production by investing irreversibly in new capital. Specific details of the game are provided below.

2.1 Production and Investment

Firms costlessly produce a flow of the good, which is non-storable, in proportion to the capital they employ in production. I will use $k_{i,t}$ to denote both a firm's stock of capital, and its production of the good, at time-t, and without loss of generality employ the index one to denote the small firm and the index two to denote the large firm. The level of production is observable.

Firms have access to the same linear, incremental investment technology. At any time a firm may increase its capital stock by an arbitrary increment $dk_{i,t}$, by purchasing new capital. The unit price of capital is constant, and normalized to one.[3] Investment is irreversible, but entails no other frictions.[4] Because investment is irreversible, it represents a credible commitment to future production.

[2] The game is a two firm version of the oligopoly game considered in Grenadier (2002), which employs the production technology, goods market, and investment technology of Leahy (1993). It also represents, conceptually, a stochastic demand version of the game studied by Spence (1979) and Fudenberg and Tirole (1983). I limit consideration to the two player game purely for the sake of expositional simplicity; the analysis extends to multi-player versions.

[3] More generally, we can allow for a time-varying cost of capital. Allowing for a stochastic capital cost allows us to include both 1) technological trends that tend to increase the productivity of new vintage capital over time, and 2) capital costs that covary positively with the demand for capital. An analysis of how this feature affects the results is left for Appendix A.1, as it both complicates the analysis, and is generally absent from the antecedent literature.

[4] In particular, I do not bound the rate of investment, and thus allow for singular control. This arises naturally as the continuous time limit of the discrete-time game in which firms chose the level of their investment each period.

Capital depreciates at the constant rate δ, so a firm's capital stock evolves according to

$$dk_{i,t} \;=\; dI_{i,t} - \delta k_{i,t} dt$$

where $I_{i,t}$ denotes the firm's aggregate investment up to time t.

2.2 Operating Profits

Firms sell their output in a competitive goods market, at the market clearing price P_t. This market clearing price is assumed to satisfy an inverse demand function of a constant elasticity form,

$$P_t \;=\; \left(\frac{X_t}{K_t}\right)^{\gamma} \tag{1}$$

where X_t is a multiplicative demand shock, $K_t = k_{1,t} + k_{2,t}$ is the aggregate quantity of the good supplied to the market at time t, $-1/\gamma$ is the price-elasticity of demand, and the demand shock follows a geometric Brownian process,

$$\frac{dX_t}{X_t} \;=\; \mu dt + \sigma dB_t$$

where $0 < \mu < r$ and σ are known constants and B_t is a standard Wiener process.[5]

[5] This formulation is equivalent to assuming that prices are set by market clearing, and that demand is time-varying at any given price, but has constant elasticity with respect to price

$$D_t \;=\; X_t P_t^{-1/\gamma}.$$

The level of the demand shock, X_t, may then be thought of as the quantity that consumers would demand if the good had unit price.

I will always assume that $\gamma < 2$, so that a symmetric duopolist cannot increase its value by destroying capital.

2.3 Firm's Objective

Firms are assumed to maximize discounted cash flows at the constant risk-free rate r, so the value of firm i is given by

$$V_i(k_{i,t}, k_{-i,t}, X_t) \;=\; \max_{\{I_{i,t+s}\}} \mathbf{E}_t\left[\int_0^\infty e^{-rs}\left(k_{i,t+s}\left(\frac{X_{t+s}}{K_{t+s}}\right)^\gamma ds - dI_{i,t+s}\right)\Big|\{I_{-i,t+s}\}\right]. \quad (2)$$

3 Equilibrium

Before considering non-trivial closed-loop strategies, our primary object of interest, it is useful to briefly consider equilibria that have previously appeared in the literature. These equilibria provide intuition that will be useful when we consider non-trivial closed-loop strategies in section 4. They also serve as useful benchmarks against which to compare our later results.

3.1 Static Market Stackelberg Game

This version of the game makes the "static market" assumptions of Spence (1979), that demand is fixed and capital does not depreciate ($\mu = \sigma = \delta = 0$), greatly simplifying the analysis. At the same time, it explicitly introduces strategic asymmetries by allowing the "leader" to move first, with the "follower" left to chase residual market opportunities. Specifically, firms are initially endowed with no capital, but choose the initial level of their stock by *sequentially* buying capital in time-zero "preplay," as in Spence (1977).

In this game the follower takes the leader's capital as given, so internalizes the price externality associated with its investment in direct proportion to its market share. The leader also internalizes the price externality of its investment in proportion to its market share, but accounts for the marginal impact of its own investment on the follower's. As a consequence, the leader only internalizes the price externality associated with its own marginal investment to the extent it impacts aggregate capital. Firms invests up to the "Stackelberg point," where each firm's marginal value of new capital equals its cost, at which point, because demand is fixed and capital does not depreciate, the game ends strategically. These facts, in conjunction with $V_i = \mathbf{E}\left[\int_0^\infty e^{-rs} k_i P ds\right] = \frac{k_i P}{r}$ and $\frac{dP}{dK} = \frac{-\gamma P}{K}$, imply that firms' marginal values of capital are given by

$$\frac{dV_F}{dk_F} = (1-\gamma s)\frac{P}{r} = 1 \tag{3}$$

$$\frac{dV_L}{dk_L} = \left(1-\gamma(1-s)\left(1+\frac{dk_F}{dk_L}\right)\right)\frac{P}{r} = 1 \tag{4}$$

where the sub-scripts F and L denote "follower" and "leader," respectively, and $s \equiv \frac{k_F}{k_L+k_F}$ is the follower's market share.

Differentiating equation (3), which implicitly defines the follower's capital reaction curve, with respect to the leader's capital stock yields the "degree of preemption," *i.e.,* the extent to which the leader's marginal investment discourages the follower's, $\frac{dk_F}{dk_L} = \frac{(1+\gamma)s-1}{2-(1+\gamma)s}$. Solving this and equations (3) and (4) simultaneously for s yields the follower's equilibrium market share in the static market Stackelberg version of the game, $s^* = \frac{2}{3+\sqrt{5-4\gamma}}$.

If demand is elastic ($\gamma < 1$) then $s^* < \frac{1}{2}$ and $\frac{dk_F}{dk_L} < 0$, *i.e.,* the leader's investment discourages the follower's investment, yielding the standard Stackelberg results. Aggregate capacity in the Stackelberg game, $K^* = \left(\frac{1-\gamma s^*}{r}\right)^{1/\gamma} X$, exceeds that in the simultaneous move symmetric Cournot equilibrium, $K_C^* = \left(\frac{1-\gamma/2}{r}\right)^{1/\gamma} X$. The capitalized value of the oligopoly rents that accrue to the leader ($V_L - k_L = (1-s^*)K^*\left(\frac{\gamma s^*}{1-\gamma s^*}\right)$) exceed those that accrue to a symmetric duopolist ($\frac{1}{2}K_C^*\left(\frac{\gamma/2}{1-\gamma/2}\right)$), which in turn exceed those that accrue to the follower ($s^* K^*\left(\frac{\gamma s^*}{1-\gamma s^*}\right)$). That is, the static market Stackelberg version of the game, when demand is elastic, delivers the Canonical intuition from the SSD literature.

If demand is inelastic ($\gamma > 1$), however, so that aggregate revenues are decreasing in aggregate supply, then firms act as if they can ignore issues of preemption. If $\gamma > 1$ then $s^* > \frac{1}{2}$ and $\frac{dk_F}{dk_L} > 0$, *i.e.,* the leader's "preemptive" investment actually encourages the follower to invest more. This renders preemptive investment unprofitable, and the firms invest as if they were playing simultaneously. The result is symmetric duopoly, with each firm internalizing half the price externality associated with its own investment, and consequently investing to half the aggregate symmetric Cournot capacity.

3.2 Perfect Competition and Monopoly

Leahy (1993) analyzes these cases under uncertainty, considering investment behavior when the industry consists of 1) a continuum of perfectly competitive firms, or 2) a single monopolistic firm. With perfect competition and the linear, incremental investment technology, the marginal and average values of capital equate. Firms consequently invest when goods prices reach the "competitive investment price threshold" $P_c^* = \Pi^{-1}$, where Π is the perpetuity factor, capitalized at the user cost of capital $r + \delta$, for a geometric Brownian process reflected from above at the reflecting barrier, and is given by

$$\Pi = \left(\tfrac{\beta-\gamma}{\beta}\right)\pi \qquad (5)$$

where $\pi = \frac{1}{r+\delta-\mu_P}$ is the perpetuity factor for the unreflected geometric Brownian price process and $\mu_P = \gamma\left(\mu + \delta - (1-\gamma)\frac{\sigma^2}{2}\right)$ is the "natural" rate of growth in goods prices (*i.e.,* the expected drift in goods prices absent new supply), and β is the positive root of the quadratic equation associated with the time-homogeneous Black-Scholes partial differential equation $\frac{\sigma^2}{2}x^2 + \left(\mu + \delta - \frac{\sigma^2}{2}\right)x - (r+\delta) = 0$.[6,7] Because firms invest whenever

[6] Letting $\zeta_t \equiv e^{\delta t}X_t / \max_{s\le t}\left\{e^{\delta s}X_s\right\}$ and $\tau \equiv \min_{s\ge 0}\{\zeta_{t+s} = 1\}$, and using the fact that ζ is a geometric Brownian process up until $t+\tau$, the perpetuity factor for a geometric Brownian process reflected from above and capitalized at $r+\delta$, the effective discount rate on capital, is given by

$$\begin{aligned}
\pi(\zeta_t) &\equiv \mathbf{E}_t\left[\int_0^\infty e^{-rs}\left(e^{-\delta s}\zeta_{t+s}\right)ds\right] \\
&= \mathbf{E}_t\left[\int_0^\tau e^{-(r+\delta)s}\zeta_{t+s}ds\right] + \mathbf{E}_t\left[e^{-(r+\delta)\tau}\right]\mathbf{E}_{t+\tau}\left[\int_0^\infty e^{-(r+\delta)s}\zeta_{t+\tau+s}ds\right] \\
&= \zeta_t\pi + \zeta_t^{\beta/\gamma}(\Pi - \pi).
\end{aligned}$$

Near the reflecting boundary $\pi(\zeta)$ is insensitive to demand shocks, *i.e.,* $\frac{d}{d\zeta}\pi(\zeta)\big|_{\zeta=1} = 0$, which implies equation (5).

[7] Note that as demand growth becomes certain (*i.e.,* as σ tends to zero), Π tends to $1/(r+\delta)$, as expected.

goods prices reach the competitive investment threshold P_c^*, and this prevents prices from rising further, prices never exceed the threshold. Below this threshold prices evolve as a geometric Brownian process.

The monopolist follows a similar strategy, but accounts fully for the price externality associated with its investment. The marginal revenue product of new capital is therefore $\frac{d}{dk}(kP) = (1-\gamma)P$: an increment dk of new capacity generates Pdk in new revenues, but by lowering goods market prices decreases revenues on its old capacity by $\gamma P dk$. The monopolist investment price threshold consequently exceeds the competitive investment price threshold, $P_m^* = P_c^*/(1-\gamma)$.

3.3 Open-Loop Cournot

Grenadier (2002) derives the open-loop Cournot equilibrium of the game (see also Baldursson (1998)).[8] While it is open-loop, a particularly natural closed-loop equilibrium, which I discuss in section 4, supports Cournot investment behavior.

For simplicity I assume here that each player is initially endowed with the same quantity of capital ($k_{1,0} = k_{2,0} = K_0/2$), and that goods prices equal the symmetric Cournot duopoly investment price threshold, $P_0 = P_C^*(1/2) \equiv P_c^*/(1-\gamma/2)$. The Cournot equilibrium consists of players committing to the capital path $k_{i,t} = (K_0/2)\, e^{-\delta t} M_t/X_0$ where $M_t = \max_{s\leq t}\{e^{\delta s}X_s\}$.

In the special case when demand is fixed ($\mu = \sigma = 0$), this equilibrium consists of a player committing to the capital path $k_{i,t} = k_{i,0}$, *i.e.,* undertaking maintenance investment that offsets the effects of depreciation. More generally, on the equilibrium path each firm invests whenever demand, relative to supply, reaches new highs. This prevents prices from ever exceeding $P_C^*(1/2)$: goods prices follow a geometric Brownian process, reflected from above at the Cournot investment price threshold, *i.e.,* $P_t = \left(e^{\delta t}X_t/M_t\right)^{\gamma} P_C^*(1/2)$.

[8] Open-loop here, in our stochastic setting, refers to the fact that the players "precommit" to their entire path of play, conditional on the path of X, at the start of the game. That is, open-loop strategies include those that allow firms to respond to the realization of nature's play.

In this equilibrium firms employ precommitment strategies. A firm's investment depends only on the level of demand, and not on its opponent's actions during the course of the game, even if its opponent plays off equilibrium.[9] Consequently, while the initial choice of the investment path is strategic, a firm's investment decisions in the course of the game are not. Because a firm's play in the course of the game has no impact on its opponents investment, each firm takes its opponents investment path as given and makes its own investment decisions accounting only for the direct impact of these decisions on its value. The marginal value of new investment capital, equal to the product of the marginal revenue product of capital and the unit value of revenue, is therefore given, because $\frac{dk_{-i}}{dk_i} = 0$ and thus $\frac{dP}{dk_i} = -\frac{\gamma P}{K}$, by

$$q_i \quad = \quad \frac{d}{dk_i}(k_i P)\Pi \quad = \quad \left(1 - \frac{\gamma k_i}{K}\right) P \Pi. \tag{6}$$

This equals one at the goods price threshold $P = P_C^*(1/2)$, and is increasing in goods market prices, so there is no incentive to deviate.

4 Closed-Loop Equilibria

In this section I consider Markov perfect equilibria of the game, in which firms can condition play on all the payoff-relevant state variables of the economy. The state variables consist, primitively, of the firms' capacities and demand, $\{k_1, k_2, X\}$.[10] It often proves convenient, however, to use as the state variables the market share of the smaller firm, aggregate capacity and prices in the goods market, $\{s \equiv \frac{k_1}{K}, K, P\}$, which represent the same information set. Then, because the firm's objective function (equation (2)) is homogeneous degree-one in k_1, k_2 and X jointly, a firm's investment strategy can be expressed solely as

[9] In particular, this equilibrium requires that each firm continues to invest on its precommitment path, even if its opponent over-invests, rendering investment on the precommitment path a negative net present value proposition. This raises concerns regarding dynamic consistency, because there is no obvious mechanism by which firms can commit to do so.

[10] The rate of investment is not included as a state variable, because the control is singular and the concept thus not well defined.

a function of s and P.

4.1 Cournot Strategies When Demand is Inelastic

As in Section 3.1, if demand is inelastic ($\gamma > 1$) then a firm can essentially ignore issues of preemption. If a monopolist would never find it optimal to add new capacity, then a firm can simply ignore the impact its investment has on its competitor's investment, and invest accounting only for the price externality associated directly with its own investment.

A firm internalizes any price externality in proportion to its market share so, ignoring issues of preemption, a firm will invest when the capitalized marginal revenue product of capital equals the cost of capital, *i.e.,* when goods prices reach the "Cournot investment price threshold" $P_C^*\left(\frac{k_i}{K}\right)$, where $P_C^*(x) = \frac{P_c^*}{1-\gamma x}$. This threshold is depicted in the top panel of Figure 1.

Now suppose that in response to a positive demand shock at the small firm's investment threshold the large firm invests to capture market share, but not enough to push prices below the small firm's investment threshold. The marginal gains to the large firm of this preemptive investment, which pushes the small firm "down its reaction curve," is the capitalized marginal revenue product of capital accounting for its impact on the small firm's investment

$$\frac{d}{dk_2}(k_2 P)\Big|_{P=P_C^*(s)} \Pi = \frac{1-(1-s)\left(1+\frac{dk_1}{dk_2}\right)\gamma}{1-\gamma s}, \tag{7}$$

where the right hand side follows from $\frac{dK}{dk_2} = 1 + \frac{dk_1}{dk_2}$, $\frac{dP}{dK} = \frac{-\gamma P}{K}$, and $\Pi P_C^*(s) = (1 - \gamma s)^{-1}$. Explicitly evaluating the right hand side of this equation requires $\frac{dk_1}{dk_2}$, which may be calculated by noting that the change in goods prices, $\frac{dP}{dk_2}\big|_{P=P_C^*(s)} = \frac{dK}{dk_2}\frac{dP}{dK}\big|_{P=P_C^*(s)}$, equals the change in the investment threshold, $\frac{dP_C^*(s)}{dk_2} = \frac{ds}{dk_2}\frac{dP_C^*(s)}{ds}$, so

$$\left(1 + \frac{dk_1}{dk_2}\right)\left(\frac{-\gamma P_C^*(s)}{K}\right) = \frac{(1-s)\left(\frac{dk_1}{dk_2}\right)-s}{K}\left(\frac{\gamma}{1-\gamma s}\right) P_C^*(s),$$

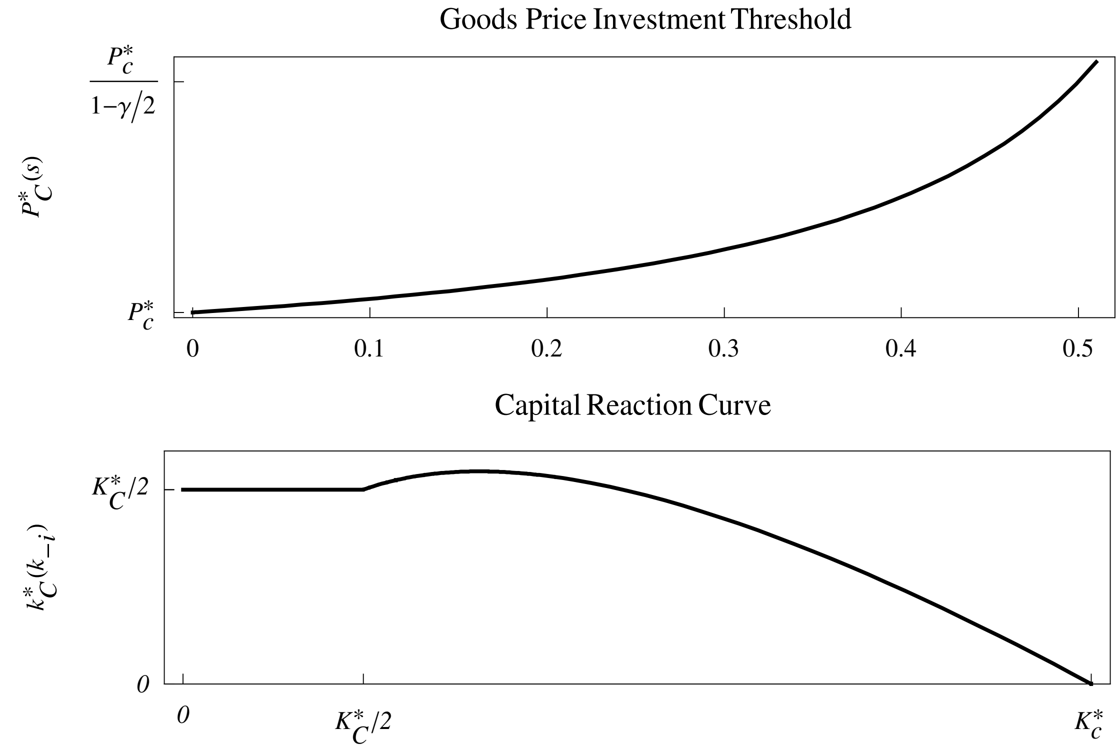

Figure 1: Cournot Investment Strategy

The top panel depicts the goods price investment strategy profile that supports shared monopoly outcome on the long-run equilibrium path, as a function of a small firm's market share. Prices are quantified as multiples of the perfectly competitive goods price investment trigger, P_c^*. The bottom panel depicts the strategy as a capital reaction curve. Capacities are quantified as multiples of the aggregate capital stock that would be built in a perfectly competitive economy, K_c^*. The figure depicts results when demand is relatively inelastic with respect to prices ($\gamma = 3/2$).

which implies that the small firm alters its investment in response to the large firm's investment by

$$\frac{dk_1}{dk_2} = \frac{(1+\gamma)s - 1}{2 - (1+\gamma)s}, \tag{8}$$

as in Section 3.1. Note that if $s = 1/2$ and $\gamma \in (1, 2)$ then this is strictly positive; if firms are symmetric and demand is inelastic, then a firm's preemptive investment does not discourage, but actually encourages, its opponent to invest.

Equations (7) and (8) together imply that at the small firm's investment boundary the marginal value of the large firm's preemptive investment, net of its cost, is $\frac{-\gamma\left((1-s)(1-2s)+\gamma s^2\right)}{(1-\gamma s)(2-(1+\gamma)s)}$, which is negative for all $s < 1/2$ when $\gamma \in (1,2)$, and hence unprofitable. The small firm can thus wait to invest at the Cournot price threshold without fear of preemption.

Note, however, that if $\gamma < 1$ then $\frac{dk_1}{dk_2}\big|_{s=1/2} < 0$, and a symmetric duopolist's preemptive investment discourages some of its competitor's investment. The marginal value of this preemptive investment consequently exceeds one, and the simple Cournot strategy cannot support an equilibrium.

4.1.1 Capital Reaction Curve

Formally, the Cournot strategy represents the continuous-time limit of the discrete-time strategy of each period investing the quantity

$$\Delta k_i \;=\; \begin{cases} (\Delta t)^a & \text{if } \frac{k_i}{K} < 1/2 \text{ and } P > P_C^*\left(\frac{k_i}{K}\right) \\ (\Delta t)^b & \text{if } \frac{k_i}{K} \geq 1/2 \text{ and } P > P_C^*\left(\frac{k_i}{K}\right) \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $0 < a < b < 1$. As the period length goes to zero, 1) the granularity of the investment disappears ($\lim_{\Delta t \to 0} (\Delta t)^a = \lim_{\Delta t \to 0} (\Delta t)^b = 0$), so there is no investment "overshooting;" 2) the small firm does all the investment ($\lim_{\Delta t \to 0} (\Delta t)^{a-b} = \infty$); and 3) firms invest on a set of measure zero (for any desired adjustment to a firm's capital stock Δk, the time it takes to undertake the investment is $\lim_{\Delta t \to 0} \frac{\Delta k}{\Delta k_i} \Delta t = 0$).

The resulting strategy can be expressed simply, using the concepts of Nonstandard analysis, as one in which 1) each firm invests at an "unlimited" investment rate whenever goods prices exceed the firm's investment price threshold, and 2) when both firms invest, the ratio of the small firm's investment rate to the large firm's investment rate is itself unlimited. Heuristically, both firms invest "infinitely fast," but the small firm invests "infinitely faster" than the large firm.

Defined this way the strategy produce well-defined outcomes. The investment price threshold $P_C^*(s) = \frac{P_C^*}{1-\gamma s}$ uniquely defines a capital reaction curve: firm i responds to its opponent's capital stock k_{-i} by choosing the capacity $\max\{k_i, k_C^*(k_{-i})\}$, where

$$k_C^*(k_{-i}) = \begin{cases} \frac{1}{2}K_C^* = \frac{1}{2}\left(P_C^*\right)^{-1/\gamma}X & \text{if } k_{-i} < \frac{1}{2}K_C^* \\ \text{The solution in } k \text{ to} & \\ \quad P_C^*\left(\frac{k}{k+k_{-i}}\right) = \left(\frac{X}{k+k_{-i}}\right)^{\gamma} & \text{if } \frac{1}{2}K_C^* < k_{-i} < K_c^* = \left(P_c^*\right)^{-1/\gamma}X \\ 0 & \text{if } k_{-i} > K_c^*. \end{cases} \tag{10}$$

This capital reaction curve, which represents an alternative description of the equilibrium investment strategy, is shown in the bottom panel of Figure 1. If, for a given level of demand, a firm's opponent has less than half the aggregate symmetric Cournot capacity, then the firm invests up to that level. If a firm's opponent has between half the Cournot capacity and the competitive capacity, then the firm invests up to the Cournot reaction curve. The fact that a symmetric duopolist's investment elicits investment from its competitor is apparent in the fact that the reaction curve is upward sloping at the point of symmetric duopoly if $\gamma > 1$. Finally, if a firm's opponent has more than the competitive capacity, then goods prices are lower than the competitive investment price threshold and the firm will not invest, even if it has no capacity.

4.1.2 Valuation

While the marginal analysis conducted above provides all the economic intuition necessary for understanding a firm's investment strategy, valuing firms explicitly allows for a "global" analysis of firms' marginal valuations of new capital. That is, after valuing firms explicitly we can verify that deviations from the proposed strategy are sub-optimal over the entire state space.

Above the investment boundary, *i.e.,* if demand is high relative to the capital firms are endowed with, then the value of a firm is given by

$$V_i^C(k_i, k_{-i}, X) = k_C^*(k_{-i})\ Q_i^C\left(\frac{k_C^*(k_2)}{k_C^*(k_2)+k_C^*(k_1)}, P_C^*\left(\frac{k_C^*(k_2)}{k_C^*(k_2)+k_C^*(k_1)}\right)\right) \tag{11}$$

where $Q_i^C(s, P)$ denotes a firm's value, relative to the replacement cost of its capital stock, as a function of s and P, and $k_C^*(k_{-i})$ is the capital reaction curve provided in equation (10).

Below the investment boundary average Q may be written as

$$Q_i^C(s, P) \;=\; \pi P + \left(\frac{P}{P_C^*(s)}\right)^{\beta/\gamma}\left(Q_i^C\left(s, P_C^*(s)\right) - \pi P_C^*(s)\right), \tag{12}$$

where $\left(\frac{P}{P_C^*(s)}\right)^{\beta/\gamma}$ is the state price, discounted at the user cost of capital $r+\delta$, for the first passage of goods prices to the investment boundary. Letting $V_i^*(s) = k_i Q_i(s, P^*(s))$ denote a firm's value at the investment threshold, the sensitivity of this value to the small firm's market share is given by

$$\frac{dV_i^*}{ds} \;=\; \frac{\frac{d\left(V_i^*-k_i\right)}{dX}\Big|_{X=X_s^*+}}{\frac{ds}{dX}\Big|_{X=X_s^*+}} + \frac{dk_i}{ds} \;=\; \frac{\frac{dV_i^*}{dX}\Big|_{X=X_s^*-}}{\frac{ds}{dX}\Big|_{X=X_s^*+}} + \frac{dk_i}{ds} \tag{13}$$

where X_s^*+ and X_s^*- signify "with respect to positive and negative demand shocks at the investment boundary," respectively, and the right hand equality follows from the smooth pasting condition. Evaluating the right hand side of equation (13), and solving the resultant differential equation subject to the relevant boundary condition, yields explicit firm valuations, provided in the following proposition. The proofs of all propositions are left for Appendix A.2.

Proposition 4.1. *If $\gamma \in (1,2)$ and firms play the Cournot strategy, then a firm's value, relative to the replacement cost of its capital stock, is given by*

$$Q_i^C(s, P) \;=\; \left(\frac{\beta}{\beta-\gamma}\right)\frac{P/P_C^*(s)}{1-\gamma s} + \left(\frac{P}{P_C^*(s)}\right)^{\beta/\gamma}\left(Q_i^*(s, 1/2, Q_C^*) - \left(\frac{\beta}{\beta-\gamma}\right)\frac{1}{1-\gamma s}\right) \tag{14}$$

where $Q_C^* = 1 + \left(\frac{\gamma/2}{1-\gamma/2}\right)\frac{\beta}{\beta-1}$,

$$\begin{aligned} Q_2^*(s, s_0, Q_{s_0}) &= \left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{1}{1-\gamma s} + \left(\tfrac{1-s_0}{1-s}\right)^{\beta}\left(\tfrac{1-\gamma s_0}{1-\gamma s}\right)^{\beta/\gamma}\left(Q_{s_0} - \left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{1}{1-\gamma s_0}\right) \\ &\quad - \left|\tfrac{\beta}{\beta-\gamma}\tfrac{\gamma}{1-\gamma}\left(\tfrac{1-\gamma}{(1-s)\gamma}\right)^{\beta}\left(\tfrac{1-\gamma}{1-\gamma s}\right)^{\beta/\gamma} B_{\frac{1-\gamma s}{1-\gamma},\frac{1-\gamma s_0}{1-\gamma}}\left(\tfrac{\beta}{\gamma},\beta\right)\right| \end{aligned} \tag{15}$$

$$\begin{aligned} Q_1^*(s, s_0, Q_{s_0}) &= \tfrac{1-s}{(\beta-1)s}\Bigg(Q_2^*(s, s_0, Q_{s_0}) - \tfrac{1-\frac{\beta s}{1-\gamma s}}{1-s} \\ &\quad - \left(\tfrac{1-s_0}{1-s}\right)^{\beta}\left(\tfrac{1-\gamma s_0}{1-\gamma s}\right)^{\beta/\gamma}\left((2-\beta)Q_{s_0} - \tfrac{1-\frac{\beta s_0}{1-\gamma s_0}}{1-s_0}\right)\Bigg), \end{aligned} \tag{16}$$

$|x|$ *denotes the modulus of x, and* $B_{x,y}(a,b)$ *is the generalized incomplete beta function*

$$B_{x,y}(a,b) \equiv \int_x^y t^{a-1}(1-t)^{b-1}dt.$$

Differentiating $k_i Q_i^C(s, P)$ with respect to k_i yields firms' marginal valuations' of new capital, provided in the following corollary.

Corollary 4.1. *Firms' marginal values of capital are given by*

$$\begin{aligned} q_2^*(s, P) &= \left(\tfrac{\beta}{\beta-\gamma}\right)\left(\tfrac{1-(1-s)\gamma}{1-\gamma s}\left(\tfrac{P}{P_C^*(s)}\right) + \tfrac{\beta-1-\gamma s}{1-\gamma s}\left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}\right) \\ &\quad - \left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}(\beta-1)Q_2^*\left(s, 1/2, Q_C^*\right) \end{aligned} \tag{17}$$

$$q_1^*(s, P) = \tfrac{\beta}{\beta-\gamma}\left(\tfrac{P}{P_C^*(s)}\right) - \tfrac{\gamma}{\beta-\gamma}\left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}. \tag{18}$$

The small firm's marginal value of capital only depends on the ratio $\frac{P}{P_C^*(s)}$, is one at the investment boundary, and less than one below the boundary. The large firm's marginal value of capital is depicted in Figure 2, both as a function of the small firm's market share at the small firm's investment boundary (*i.e.,* when $P = P_C^*(s)$; left panel), and as a function of both the small firm's market share and goods prices relative to the small firm's investment threshold (s and $p = P/P_C^*(s)$; right panel). The large firm's marginal value of

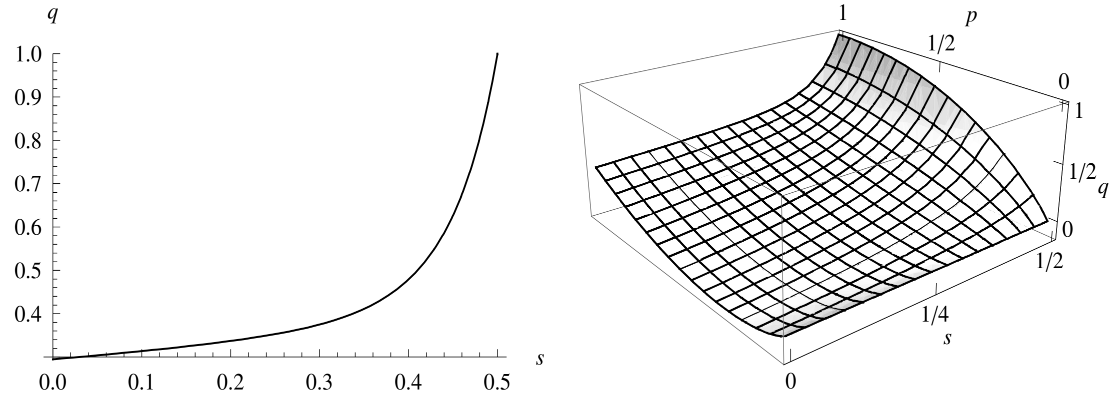

Figure 2: Large Firm's Marginal Value of Capital

The left panel shows the large firm's marginal value of capital at the small firm's investment threshold ($P_C^*(s)$), as a function of the small firm's market share (s). The right panel shows the large firm's marginal value of capital as function of both the small firm's market share and goods prices relative to the small firm's investment threshold (s and $p = P/P_C^*(s)$, respectively). The figure shows results when demand is relatively inelastic with respect to prices ($\gamma = 3/2$) and $\beta = 2$.

capital is always less than one if $s < 1/2$ or $\frac{P}{P_C^*(s)} < 1$, and equals one only if $s = 1/2$ and $P = P_C^*(1/2)$, so the large firm never has an incentive to deviate. The strategy described above consequently supports a dynamically consistent equilibrium.

4.1.3 Time to symmetric duopoly

Because the large firm cedes market share over time, undertaking no investment until the small firm has captured half the market, market symmetry is achieved at $\tau = \min_{t>0}\{K_t = 2(1-s)e^{-\delta t}K_0\}$, at which point $P_\tau = \left(e^{\delta\tau}X_\tau/2(1-s)K_0\right)^\gamma = P_C^*(1/2)$. This may be written, in terms of demand, as $\left(e^{\delta\tau}X_\tau/X_0\right)^\gamma = 2^\gamma(1-s)^\gamma P_C^*(1/2)/P_0$, which using standard results for stopping times of Brownian processes implies that

$$\mathbf{E}[\tau] = \frac{\ln\left(2(1-s)\left(\frac{P_C^*(1/2)}{P_0}\right)^{1/\gamma}\right)}{\mu+\delta-\frac{\sigma^2}{2}} \tag{19}$$

$$\mathrm{var}[\tau] = \frac{\mathbf{E}[\tau]}{\left(\frac{\mu+\delta}{\sigma}-\frac{\sigma}{2}\right)^2}. \tag{20}$$

These equations will also hold for the equilibria considered in Sections 4.2 and 4.3.

4.2 Shared Monopoly

Another Markov perfect equilibrium of the game, perhaps even simpler than the Cournot equilibrium, is the one in which firms split the market, together investing like a monopolist on the long-run equilibrium path. In this equilibrium, as in the case of true monopoly, investment only occurs (except, perhaps, at the start of the game) if demand is sufficiently elastic that aggregate revenues are increasing in aggregate capacity (*i.e.,* $\gamma < 1$). I therefore restrict attention to this case.[11]

Each firm's strategy in the "shared monopoly" equilibrium consists of "playing Cournot" whenever the firm has less than half the market, and investing at the monopoly investment price threshold whenever the firm has at least half the market, *i.e.,* investing at the shared monopoly price threshold

$$P^*_{sm}\left(\frac{k_i}{K}\right) = \begin{cases} P^*_C\left(\frac{k_i}{K}\right) & \text{if } \frac{k_i}{K} < 1/2 \\ P^*_m \equiv P^*_c/(1-\gamma) & \text{if} \left(\frac{k_i}{K}\right) \geq 1/2. \end{cases} \tag{21}$$

The shared monopoly goods price investment strategy profile is shown in the top panel of Figure 3.

The intuition for the strategy is again basically Cournot. The firm invests whenever the marginal revenue product of capital is at least as great as its opportunity cost. With a market share less than one half this occurs at the Cournot reaction curve. With a market share equal to one half a firm only internalizes half of the price externality associated with new investment, but understands that for every unit of capacity it adds it elicits an equal

[11] I will also assume that the demand does not grow "too" slowly. This simplifies the analysis, by assuring that preemptive investment is always suboptimal. In particular, I assume that $\beta < \beta^*(\gamma)$, where $1/\beta^*(\gamma) \approx 0.1747(1-\gamma)$. In terms of demand growth, the restriction implies $\mu > \frac{r+\delta}{\beta^*(\gamma)} - \delta + (1-\beta^*(\gamma))\frac{\sigma^2}{2}$. Note that in the special case of certain growth and no depreciation ($\delta = \sigma = 0$) this implies $\mu \gtrsim 0.1747(1-\gamma)r$. The derivation of the parameter restriction, and a discussion of the strategy that supports the shared monopoly outcome when the parameter restriction is violated, are left for Appendix A.3.

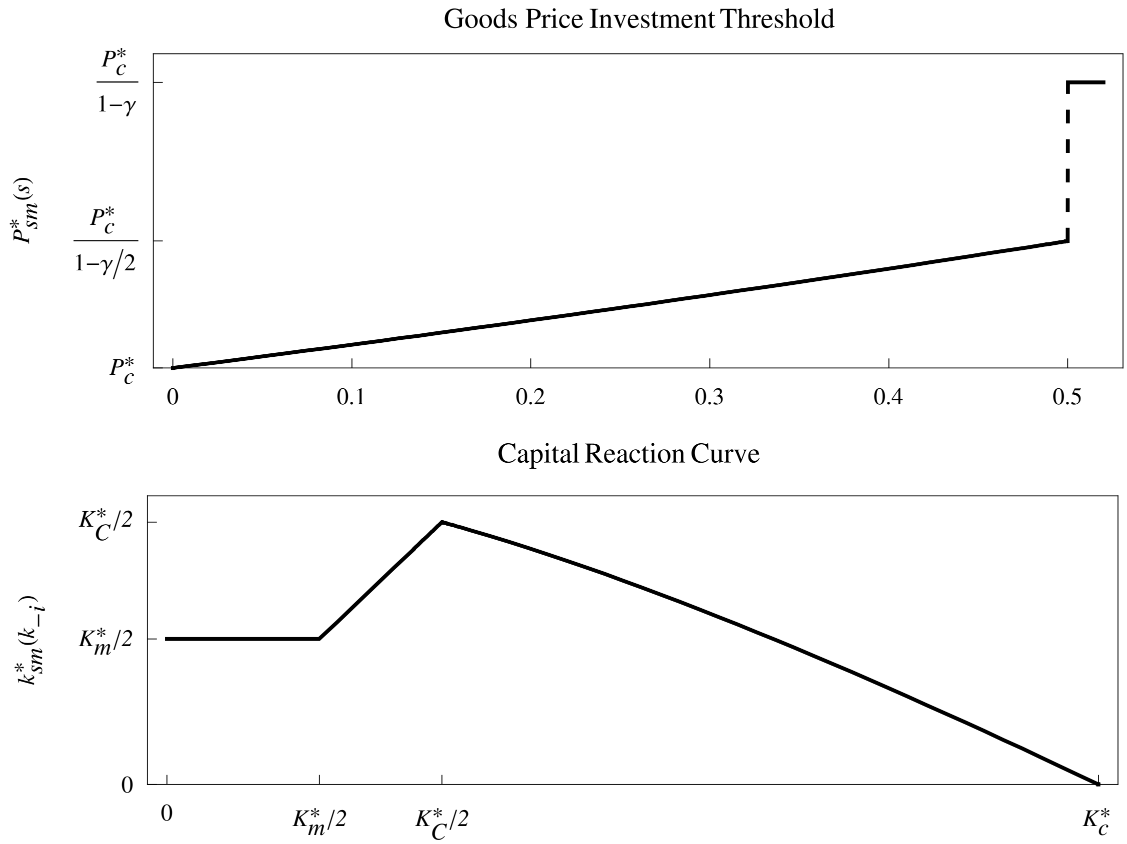

Figure 3: Shared Monopoly Investment Strategy

The top panel depicts a firm's investment strategy profile, in goods market prices and as a function of the firm's market share, which supports the shared monopoly outcome on the long-run equilibrium path. Prices are quantified as multiples of the perfectly competitive goods price investment trigger, P_c^*. The bottom panel depicts the strategy as a capital reaction curve. Capacities are quantified as multiples of the aggregate capital stock that would be built in a perfectly competitive economy, K_c^*, while K_m^* is the capital stock that would by built a monopolist and K_C^* is the aggregate capacity in the symmetric Cournot duopoly. The figure shows results when demand is highly elastic with respect to prices ($\gamma = 1/5$).

investment from the other player. Internalizing half of the price externality associated with this double investment supports the monopoly path for goods prices and aggregate capacity.

Formally, this strategy represents the continuous-time limit of the discrete-time strategy of each period investing the quantity

$$\Delta k_i = \begin{cases} \min\{(\Delta t)^a, k_{-i} - k_i\} & \text{if } \frac{k_i}{K} < 1/2 \text{ and } P_C^*\left(\frac{k_i}{K}\right) < P \leq P_m^* \\ \min\{(\Delta t)^a, k_{-i} - k_i + (\Delta t)^b\} & \text{if } \frac{k_i}{K} < 1/2 \text{ and } P > P_m^* \\ (\Delta t)^b & \text{if } \frac{k_i}{K} \geq 1/2 \text{ and } P > P_m^* \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

where $0 < a < b < 1$. Defined this way the shared monopoly price threshold implies the shared monopoly capital reaction curve

$$k_{sm}^*(k_{-i}) = \begin{cases} \frac{1}{2}K_m^* = \frac{1}{2}\left(P_m^*\right)^{-1/\gamma}X & \text{if } k_{-i} < \frac{1}{2}K_m^* \\ k_{-i} & \text{if } \frac{1}{2}K_m^* < k_{-i} < \frac{1}{2}K_C^* \\ k_C^*(k_{-i}) & \text{if } k_{-i} > \frac{1}{2}K_C^* \end{cases} \quad (23)$$

where $K_C^* = \left(P_C^*\right)^{-1/\gamma}X$, and $k_C^*(k_{-i})$ is as defined in equation (10). This capital reaction curve is shown in the bottom panel of Figure 3. If, for a given level of demand, a firm's opponent has less than half the monopoly capacity, then the firm invests up to that level (this gets firms to the left hand kink in the capital reaction curve, and corresponds to the upper kink in the goods price investment threshold). If a firm's opponent has between half the monopoly capacity and half the aggregate symmetric Cournot capacity, then the firm invests up to the level of its opponent (the upward sloping segment between the two kinks in the capital reaction curve, which corresponds to the vertical segment between the two kinks in the goods price investment threshold). If a firm's opponent has between half the Cournot capacity and the competitive capacity, then the firm invests up to the Cournot reaction curve (the right hand curved segment in the capital reaction curve, which corresponds to the upward sloping curve in the goods price investment threshold). If a firm's opponent has more than the competitive capacity, then the firm will not invest.

4.2.1 Valuation

Above the investment boundary, *i.e.,* if demand is high relative to the capital firms are endowed with, the value of a firm is given by

$$V_i^{sm}(k_i, k_{-i}, X) = k_{sm}^*(k_{-i})\ Q_i^{sm}\left(\tfrac{k_{sm}^*(k_2)}{k_{sm}^*(k_2)+k_{sm}^*(k_1)}, \left(\tfrac{X}{k_{sm}^*(k_2)+k_{sm}^*(k_1)}\right)^{\gamma}\right) \tag{24}$$

where $Q_i^{sm}(s, P)$ denotes a firm's value, relative to the replacement cost of its capital stock, as a function of s and P, and $k_{sm}^*(k_{-i})$ is the shared monopoly capital reaction curve provided in equation (23). At and below the goods price investment price threshold, firms may be valued explicitly using the same methodology employed in the previous section. Results of this valuation are provided in the following proposition.

Proposition 4.2. *If firms play the shared monopoly strategy, then a firm's value, relative to the replacement cost of its capital stock, is given by*

$$Q_i^{sm}(s, P) = \left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{P/P_C^*(s)}{1-\gamma s} + \left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}\left(Q_i^*(s, \tfrac{1}{2}, Q_{sm}^*) - \left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{1}{1-\gamma s}\right) \tag{25}$$

where $Q_{sm}^* = \left(\frac{1}{1-\gamma/2}\right)\left(\frac{\beta}{\beta-\gamma}\right) + \left(\frac{1-\gamma}{1-\gamma/2}\right)^{\beta/\gamma}\frac{\gamma}{(\beta-1)(\beta-\gamma)}$ *and* $Q_i^*\left(s, s_0, Q_{s_0}\right)$ *is as defined in Proposition 4.1 (equations (15) and (16)).*

Differentiating $k_i Q_i^{sm}(s, P)$ with respect to k_i yields firms' marginal valuations' of new capital, provided in the following corollary.

Corollary 4.2. *If* $P \le P_C^*(s)$ *then firms' marginal values of capital are given by*

$$\begin{aligned} q_2^*(s, P) \quad &= \quad \left(\tfrac{\beta}{\beta-\gamma}\right)\left(\tfrac{1-(1-s)\gamma}{1-\gamma s}\left(\tfrac{P}{P_C^*(s)}\right) + \tfrac{\beta-1-\gamma s}{1-\gamma s}\left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}\right) \\ &\qquad - \left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}(\beta-1)\,Q_2^*\left(s, 1/2, Q_{sm}^*\right) \end{aligned} \tag{26}$$

$$q_1^*(s, P) \quad = \quad \tfrac{\beta}{\beta-\gamma}\left(\tfrac{P}{P_C^*(s)}\right) - \tfrac{\gamma}{\beta-\gamma}\left(\tfrac{P}{P_C^*(s)}\right)^{\beta/\gamma}. \tag{27}$$

If $s = 1/2$ *and* $P_C^*(1/2) < P \le P_m^*$ *then a firm's marginal value of capital is* $(1-\gamma)P/P_c^*$.

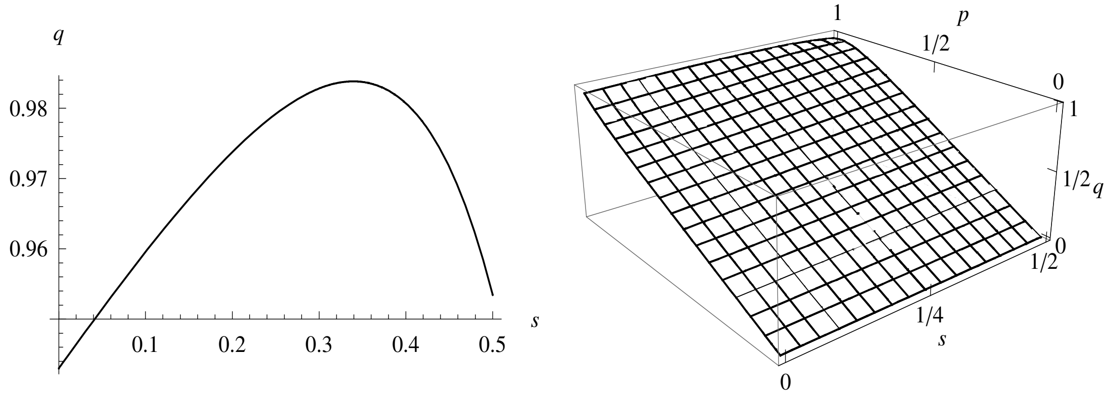

Figure 4: Large Firm's Marginal Value of Capital
The left panel shows the large firm's marginal value of capital at the small firm's investment threshold ($P^*_{sm}(s)$), as a function of the small firm's market share (s). The right panel shows the large firm's marginal value of capital as a function of both the small firm's market share and goods prices relative to the small firm's investment threshold (s and $p = P/P^*_{sm}(s)$, respectively). The figure shows results when demand is highly elastic with respect to prices ($\gamma = 1/5$) and $\beta = 2$.

The small firm's marginal value of capital again only depends on the ratio $\frac{P}{P^*_{sm}(s)}$, is less than one below the investment boundary, and one at the investment boundary. The large firm's marginal value of capital is depicted in Figure 4, both as a function of the small firm's market share at the investment boundary (*i.e.,* when $P = P^*_{sm}(s)$; left panel), and as a function of both the small firm's market share and goods prices relative to the small firm's investment threshold (s and $p = P/P^*_{sm}(s)$; right panel). It is always less than one if $s < 1/2$ or $\frac{P}{P^*_{sm}(s)} < 1$. It equals one only if $s = 1/2$ and $P = P^*_m$, so the firm has no incentive to deviate.

4.2.2 Issues with the Equilibrium

Agents support the shared monopoly equilibrium using an "active" Markov punishment mechanism. If the firms have equal capacity and one firm deviates by investing when prices are lower than the monopoly investment trigger, it elicits an *immediate* "tit-for-tat" response of equal magnitude. As a consequence, if firms followed the strategy profile given in Figure 3 but observed their opponent's capacity with an arbitrarily small error, then firms at the monopoly price threshold would instantaneously invest down to the Cournot duopoly price level. Perceived deviations from equal market shares would elicit an investment response from the "smaller" agent, which would themselves elicit a response from the other player. Firms would consequently immediately add enough capacity to reduce prices to the Cournot level, and without an effective punishment mechanism to support the monopoly investment price threshold when market shares are equal, the rest of the strategy profile does not support an equilibrium strategy. The equilibrium is thus unstable to the inclusion of *any* noise in the observed state variables, *i.e.,* with imperfect observables the equilibrium is not public perfect, in the sense of Sannikov (2007).

4.3 Preemptive Preemption and Cournot Outcomes

An alternative to the shared monopoly strategy, robust to the issue of public perfection, is a class of strategies that involves "preempting preemption." In this class of strategies the small firm, recognizing that the large firm will preempt its investment whenever it is profitable for it to do so, responds by preempting the large firm's preemptive investment.

Deriving the preemptive preemption strategies requires only that we recognize that the small agent will always preempt the larger agent's preemptive investment, simply because she can. The shadow price of capital is always lower for the small agent, because she internalizes less of the price externality associated with new investment.

Because the small firm preempts the large firm's preemptive investment, we need to calculate the price level at which the large firm would undertake preemptive investment. The intuition for the large firm's preemption strategy is again Cournot. The larger agent would preempt the smaller firm's investment if the large firm's marginal value of capital, accounting for the price externality and the effects of preemption on this price externality, exceeds the cost of capital. Its indifference condition on investment, letting $P_p^*(s)$ denote the goods price preemption profile, is therefore

$$\left(1-(1-s)\left(1+\frac{dk_1}{dk_2}\right)\gamma\right)\Pi P_p^*(s) \;=\; 1. \tag{28}$$

Note that $(1-s)\left(1+\frac{dk_1}{dk_2}\right)\gamma$ simply represents the effective price externality internalized by the large firm: the large firm has a market share of $1-s$, net new aggregate capacity that results from the firm adding an increment dk_2 of new capacity is $\left(1+\frac{dk_1}{dk_2}\right)dk_2$, and γ is the inverse price-elasticity of demand.

The previous equation, in conjunction with the fact that the large firm will only undertake preemptive investment at, and not below, the preemption price profile, so $\frac{dP_p^*}{dk_2}\big|_{P=P_p^*} = \frac{dP}{dk_2}\big|_{P=P_p^*}$, implies a differential equation that, subject to the appropriate boundary condition, specifies the preemption profile. This preemption profile, essentially an incentive compatibility constraint on the large firm not investing, is described in the following proposition.

Proposition 4.3. *The preemption profile satisfies*

$$\frac{dP_p^*(s)}{ds} \;=\; \frac{-\gamma P_p^*(s)}{1-s}\left(\frac{P_p^*(s)-P_c^*}{(1-\gamma)P_p^*(s)-P_c^*}\right). \tag{29}$$

Equation (29) describes a one-parameter family of profiles, corresponding to the choice of boundary condition. The boundary condition $P_p^*(1/2) = P_c^*$ corresponds to a strategy that yields perfectly competitive outcomes, in which firms always invest at the competitive price threshold P_c^*, similar to that suggested by Back and Paulson (2009). Boundary conditions in excess of the symmetric Cournot duopoly price threshold, $P_p^*(1/2) > P_C^*(1/2)$, correspond to profiles that fail to satisfy the Markov restriction. The boundary condition $P_p^*(1/2) = P_C^*(1/2)$ corresponds to the profile that yields the Pareto dominant Markov perfect strategy in the class of preemptive preemption strategies. I consequently focus on this case; the rest of this class is discussed in greater detail in Appendix A.4.

The preemptive preemption investment strategy consists of investing at the minimum of the price levels given by the Cournot reaction curve and the preemption curve described in Proposition 4.3, $P^*_{pp}(s) \equiv \min\left\{P^*_C(s), P^*_p(s)\right\}$.[12] The preemptive preemption goods price investment strategy profile is shown in the top panel of Figure 5, where the critical market share s^*_{pp} in the figure is defined implicitly as the solution in s to $P^*_p(s) = P^*_C(s)$. Formally this strategy represents the continuous-time limit of the discrete-time strategy of each period investing the quantity

$$\Delta k_i \quad = \quad \begin{cases} (\Delta t)^a & \text{if } \frac{k_i}{K} < 1/2 \text{ and } P > P^*_{pp}(s) \\ (\Delta t)^b & \text{if } \frac{k_i}{K} \geq 1/2 \text{ and } P > P^*_p(s) \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

where $0 < a < b < 1$.

The preemption curve in the strategy profile can be understood as a Markov mechanism by which the small firm either 1) punishes the large firm if it invests, or 2) rewards the large firm for not investing. Viewed “on the way down,” the preemption curve describes the minimal punishment strategy that supports Counrot outcomes. If the large firm deviates by investing and capturing market share, the small firm lowers the price at which it will invest next just enough so that the large firm is no better off for its deviation. Viewed “on the way up,” the preemption curve describes the minimal reward strategy that supports Cournot outcomes. The small firm “bribes” the large firm into not investing, by allowing prices to rise sufficiently quickly, even as the small firm continues to invest and capture market share, such that the large firm is no better off if it invests. The curve is just steep enough at every point so that the large firm is indifferent between investing and ceding market share to the smaller firm.

[12] Note that the simple Cournot strategy described in section 4.1 is nested within this strategy. If $\gamma > 1$, then $P^*_C(s) \leq P^*_p(s)$ for all $s \in [0, 1/2]$, so $P^*_{pp}(s) = P^*_C(s)$.

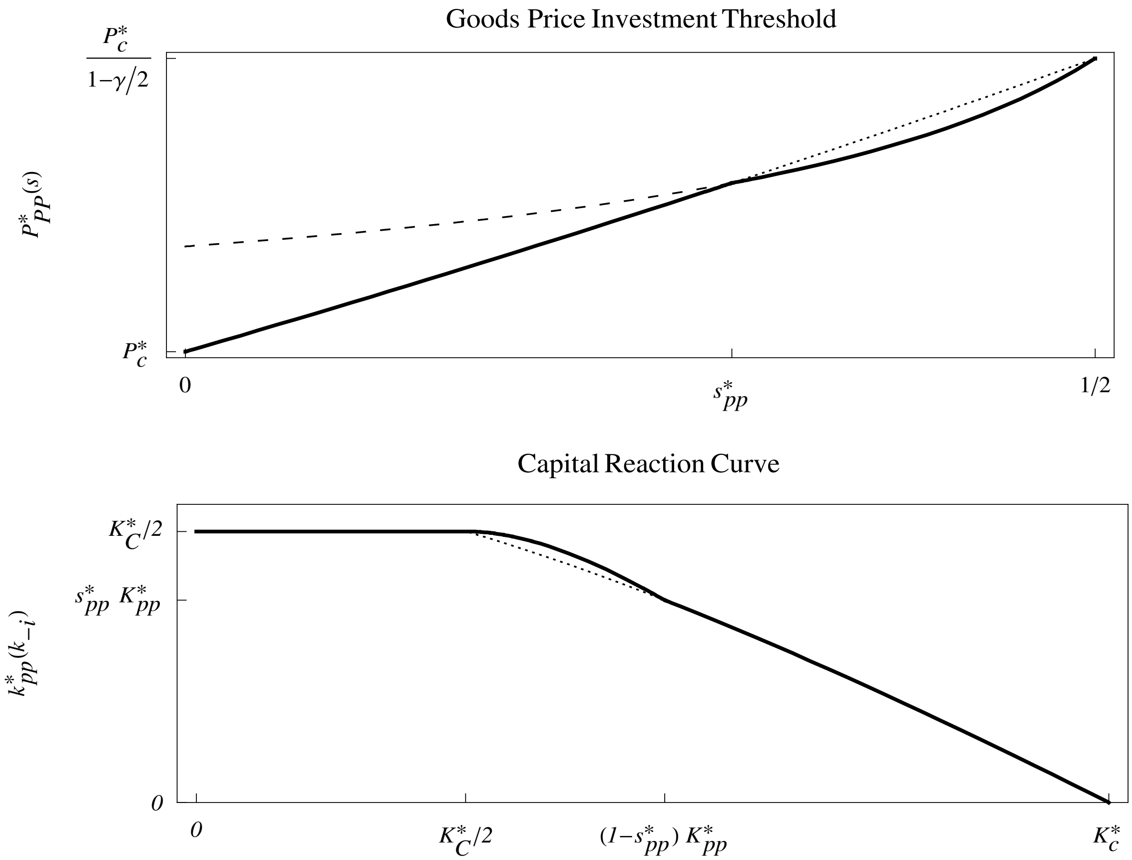

Figure 5: Preemptive Preemption Investment Strategy

The top panel depicts the the investment strategy profile, in goods market prices and as a function of the firm's market share, which supports the shared monopoly outcome on the long-run equilibrium path. Prices are quantified as a multiple of the perfectly competitive goods price investment trigger, P_c^*, and s_{pp}^* is the critical market share beneath which the small firm plays the Cournot strategy without fear of preemption. The bottom panel depicts the strategy as a capital reaction curve. Capacities are quantified as a multiple of the aggregate capital stock that would by built in a perfectly competitive economy, K_c^*, and K_C^* is the aggregate capacity in the symmetric Cournot duopoly. The figure shows results when demand is highly elastic with respect to prices ($\gamma = 1/5$).

The preemptive preemption investment price threshold implies the capital reaction curve

$$k_{pp}^*(k_{-i}) = \begin{cases} \frac{1}{2}K_C^* & \text{if } k_{-i} < \frac{1}{2}K_C^* \\ \text{The solution in } k \text{ to} & \\ \quad P_{pp}^*\left(\frac{k}{k+k_{-i}}\right) = \left(\frac{X}{k+k_{-i}}\right)^{\gamma} & \text{if } \frac{1}{2}K_C^* < k_{-i} < \left(1-s_{pp}^*\right)K_{pp}^* \\ k_C^*(k_{-i}) & \text{if } k_{-i} > \left(1-s_{pp}^*\right)K_{pp}^* \end{cases} \tag{31}$$

where $K_{pp}^* = P_C^*\left(s_{pp}^*\right)^{-1/\gamma} X$. This capital reaction curve is shown in the bottom panel of Figure 5. The key feature of this reaction curve is that it is "flat" at the symmetric Cournot point, $\frac{dk_{pp}^*(k_{-i})}{dk_{-i}}\Big|_{k_{-i}=K_C^*/2} = 0$, so a firm's investment is insensitive to its competitor's capital stock. At this point a firm consequently makes its marginal investment decision accounting only for the price externality associated directly with its own investment, because these decisions have no impact on its opponent's investment, and the result is Cournot outcomes.

This Cournot behavior results, counter to the standard Stackelberg intuition, precisely because capital represents a strong commitment to future production. While in a static setting (*e.g.,* in the Stackelberg model) the commitment power of irreversible investment allows a "leader" to credibly preempt a "follower's" investment, in a dynamic setting it provides a means for the follower to credibly preempt the leader's preemptive investment, yielding Cournot outcomes. This can be framed, perversely, using the Stackelberg intuition: the advantageous position that a small market share confers on the "follower" allows her to play the role of "Stackelberg leader" in the competition for *new* capacity.

Similar results can be observed in the linear-differential oligopoly game studied by Hanig (1987) and Reynolds (1987). In this game firms face quadratic adjustment costs, so capital is reversible and only provides a limited commitment to future production, which is increasing in the magnitude of the adjustment costs.[13] Firms' engage in simultaneous Stackelberg behavior, and their capital stocks in the closed-loop equilibrium exceed their levels in the open-loop Cournot equilibrium, a fact that has garnered most of the attention given to their model. Essentially no attention has been paid, however, to the fact that when capital represents a strong commitment to future production, then a firm's capital stock can provided a strong deterrent to its opponent's preemptive investment. Firms consequently develop less "extra" capacity when adjustment costs are high. In the limit, as the adjust-

[13] Hanig shows that the firms' capital stocks tend to their Stackelberg levels as adjustment costs become high for one firm and low for the other, with the high adjustment cost firm playing the leader. Generally, if only one firm possesses a mechanism for credibly committing to its level of production, then this firm produces like the Stackelberg leader confident that other firms, presented with this production as a *fait accompli,* will produce on their Cournot reaction curve.

ment costs tend to infinity, firms' behavior in the steady state equilibrium tends to Cournot (*i.e.,* the ratio of aggregate capacity in the closed-loop equilibrium to that in the open-loop equilibrium tends to one).

4.3.1 Valuation

Above the investment boundary, *i.e.,* if demand is high relative to the capital firms are endowed with at the start of the game, the value of a firm is given by

$$V_i^{pp}(k_i, k_{-i}, X) = k_{pp}^*(k_{-i})\ Q_i^{pp}\left(\tfrac{k_{pp}^*(k_2)}{k_{pp}^*(k_2)+k_{pp}^*(k_1)}, \left(\tfrac{X}{k_{pp}^*(k_2)+k_{pp}^*(k_1)}\right)^{\gamma}\right) \tag{32}$$

where $Q_i^{pp}(s, P)$ denotes a firm's value, relative to the replacement cost of its capital stock, as a function of s and P, and $k_{pp}^*(k_{-i})$ is the capital reaction curve provided in equation (31).

To calculate the value of the large firm at $P_p^*(s)$, note that it could maintain its market share of $1-s$, if it were profitable to do so, by investing "at" (*i.e.,* just below) the small firm's investment threshold. In order to discourage this preemptive investment, the value of the large firm, net of investment costs, must be non-decreasing on the small firm's investment price profile as the small firm invests and captures market share. After demand has risen sufficiently so that the small firm has captured half the market, the value of the large firm must consequently weakly exceed the value it would have had if it had invested to maintain its initial, larger market share, *i.e.,*

$$(1-s)K_s^*\left(Q_i^{pp}\big(s, P_p^*(s)\big) - 1\right) \quad \leq \quad \tfrac{1}{2}K_{1/2}^*\left(Q_i^{pp}\big(\tfrac{1}{2}, P_p^*\big(\tfrac{1}{2}\big)\big) - 1\right)$$

where aggregate capacity in the two cases (K_s^* and $K_{1/2}^*$, respectively) are related by ${K_s^*}^{\gamma} P_p^*(s) = {K_{1/2}^*}^{\gamma} P_p^*(1/2)$.

If $P_p^*(s)$ describes the large firm's indifference curve, then the left and right hand sides of the previous equation equate. The value of the large firm may then be calculated by assuming that it always invests to maintain its market share, so

$$(1-s)K_s^*\left(\tfrac{P_p^*(s)}{P_c^*}-1\right)\tfrac{\beta}{\beta-1} \;=\; \tfrac{1}{2}K_{1/2}^*\left(\tfrac{P_p^*(1/2)}{P_c^*}-1\right)\tfrac{\beta}{\beta-1}.$$

The previous equation, in conjunction with ${K_s^*}^{\gamma} P_p^*(s) = {K_{1/2}^*}^{\gamma} P_p^*(1/2)$, provides an alternative characterization of the preemption strategy profile, implicitly defined for each $s \in [0, 1/2]$ by

$$\frac{(1-s)\left(P_p^*(s)-P_c^*\right)}{P_p^*(s)^{1/\gamma}} \;=\; \frac{\frac{1}{2}\left(P_p^*(1/2)-P_c^*\right)}{P_p^*(1/2)^{1/\gamma}}. \tag{33}$$

Note that differentiating the previous equation with respect to s and solving the result for $\frac{dP_p^*}{ds}$ yields the characterization provided in Proposition 4.3 (equation (29)).

The fact that the large firm is indifferent, at the preemption threshold, between investing to maintain its market share and ceding market share to the small firm, makes explicit valuation of the large firm simple when s exceeds the critical market share s_{pp}^*. The small firm may be valued then by noting that equation (13), together with $\frac{ds}{dX}\big|_{X=X_s^*+} = \frac{dV_2^*}{dX}\big|_{X=X_s^*+} / \frac{dV_2^*}{ds}$ and the smooth pasting condition on V_2^*, implies that

$$\frac{dV_1^*}{ds} \;=\; \left(\frac{\frac{dV_1^*}{dX}\big|_{X=X_s^*-}}{\frac{dV_2^*}{dX}\big|_{X=X_s^*-}}\right)\frac{dV_2^*}{ds}+\frac{dk_1}{ds}. \tag{34}$$

Evaluating the right hand side of the previous equation yields a differential equation that describes the small firm's value above the critical market share s_{pp}^*. Below the critical market share both firms may be valued using the techniques employed in the previous two sections. Complete valuations are provided in the following proposition.

Proposition 4.4. *If firms play the preemptive preemption strategy, then a firm's values, relative to the replacement cost of its capital stocks, is given by*

$$Q_i^{pp}(s,P) \;=\; \left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{P}{P_c^*}+\left(\tfrac{P}{P_{pp}^*(s)}\right)^{\beta/\gamma}\left(Q_i^{pp}\big(s,P_{pp}^*(s)\big)-\left(\tfrac{\beta}{\beta-\gamma}\right)\tfrac{P_{pp}^*(s)}{P_c^*}\right) \tag{35}$$

where

$$Q^2_{pp}\big(s, P^*_{pp}(s)\big) \;=\; \begin{cases} 1+\left(\frac{P^*_{pp}(s)}{P^*_c}-1\right)\frac{\beta}{\beta-1} & \textit{if } s \geq s^*_{pp} \\ Q^*_2\Big(s, s^*_{pp}, 1+\left(\frac{\gamma s^*_{pp}}{1-\gamma s^*_{pp}}\right)\frac{\beta}{\beta-1}\Big) & \textit{if } s < s^*_{pp} \end{cases} \tag{36}$$

$$Q^1_{pp}\big(s, P^*_{pp}(s)\big) \;=\; \begin{cases} \left(\frac{1-s}{s}\right)\nu(s) & \textit{if } s \geq s^*_{pp} \\ Q^*_1\big(s, s^*_{pp}, Q^1_{pp}\big(s^*_{pp}, P^*_{pp}\big(s^*_{pp}\big)\big)\big) & \textit{if } s < s^*_{pp}, \end{cases} \tag{37}$$

where $Q^*_i\big(s, s_0, Q_{s_0}\big)$ is again as defined in Proposition 4.1 (equations (15) and (16)), and $\nu(s)$ satisfies the differential equation

$$\frac{d\nu(s)}{ds} \;=\; \frac{\gamma\beta}{1-s}\left(\frac{P^*_{pp}(s)}{P^*_c-(1-\gamma)P^*_{pp}(s)}\right)\left(\nu(s)-\frac{s}{1-s}\left(\frac{P^*_{pp}(s)}{P^*_c}\right)\right)+\frac{1}{(1-s)^2} \tag{38}$$

subject to the boundary condition $\nu(1/2) = 1+\left(\frac{\gamma/2}{1-\gamma/2}\right)\frac{\beta}{\beta-1}$.

Differentiating $k_i Q^{pp}_i(s,P)$ with respect to k_i yields the following corollary.

Corollary 4.3. *Firms' marginal values of capital are given by*

$$q^*_2(s,P) \;=\; \begin{cases} \frac{\beta}{\beta-\gamma}\left(\frac{1-\gamma(1-s)}{1-\gamma s}\left(\frac{P}{P^*_{pp}(s)}\right)-\frac{\gamma s}{1-\gamma s}\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma}\right) & \textit{if } s < s^*_{pp} \\ \quad -(\beta-1)\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma}\left(Q^2_{pp}\big(s, P^*_{pp}(s)\big)-\frac{\left(\frac{\beta}{\beta-\gamma}\right)}{1-\gamma s}\right) & \\ \frac{\beta(1-\gamma(1-s))}{\beta-\gamma}\left(\frac{P}{P^*_{pp}(s)}-\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma}\right)\frac{P^*_{pp}(s)}{P^*_c}+\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma} & \textit{if } s > s^*_{pp} \end{cases} \tag{39}$$

$$q^*_1(s,P) \;=\; \begin{cases} \frac{\beta}{\beta-\gamma}\left(\frac{P}{P^*_{pp}(s)}\right)-\frac{\gamma}{\beta-\gamma}\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma} & \textit{if } s < s^*_{pp} \\ \frac{\beta(1-\gamma s)}{\beta-\gamma}\left(\frac{P}{P^*_{pp}(s)}-\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma}\right)\frac{P^*_{pp}(s)}{P^*_c}+\left(\frac{P}{P^*_{pp}(s)}\right)^{\beta/\gamma} & \textit{if } s > s^*_{pp}. \end{cases} \tag{40}$$

The small firm's marginal value of capital is less than one below the investment boundary, and equals one at the boundary. The large firm's marginal value of capital, which is

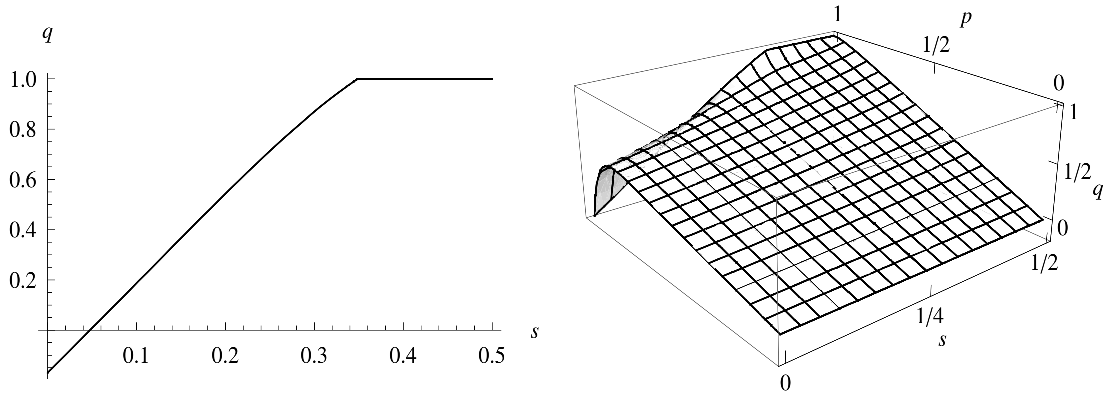

Figure 6: Marginal Values of Capital

The left panel shows the large firm's marginal value of capital at the small firm's investment threshold ($P_{pp}^*(s)$), as a function of the small firm's market share (s). The right panel shows the large firm's marginal value of capital as a function of both of the small firm's market share and goods prices relative to the small firm's investment threshold (s and $p = P/P_{pp}^*(s)$, respectively). The figure shows results when demand is highly elastic with respect to prices ($\gamma = 1/5$) and $\beta = 2$.

depicted in Figure 6, is always less than one if $s < s_{pp}^*$ or $\frac{P}{P_C^*(s)} < 1$, and equals one if $s > s_{pp}^*$ and $P = P_{pp}^*(s)$.

5 Implications for Stackelberg Behavior

We now consider implications for Stackelberg behavior, by again considering the "Stackelberg version" of the game, in which strategic asymmetries are introduced explicitly by allowing the "leader" to move first, with the "follower" left to chase residual market opportunities. Specifically, in this version of the game firms are initially endowed with no capital, and choose the initial level of their stock by sequentially buying capital in time-zero preplay. Section 3.1 analyzed this game under the static market assumptions of fixed demand and non-depreciatingn capital ($\mu = \sigma = \delta = 0$), and derived results consistent with the canonical SSD intuition, with the leader investing beyond the capacity of a symmetric duopolist in order to partially foreclose the follower's market opportunities through the price externality channel.

The Stackelberg behavior in the static market Stackelberg version of the game is predicated on there being no "next" investment opportunity after firms have invested up to their Cournot reaction cures, and thus depends on markets being *completely* static. If demand grows at all, or capital depreciates at any rate, the nature of firms' dynamic programming problem is dramatically altered. Stackelberg behavior then does not represent a solution, as the leader's preemptive investment is unprofitable in dynamically consistent equilibria of the game.

When demand grows or capital depreciates, the equilibria presented in Section 4 represent solutions to the continuation game, after firms have bought their initial capital stocks. In none of these equilibria can the leader profitably buy "preemptive" capacity in preplay. In the case of the simple Cournot strategy when demand is inelastic this is because preemption is simply unprofitable, while in the cases of the shared monopoly and preemptive preemption strategies it is because a firm's opponents' response to preemptive investment is designed to make preemptive investment unprofitable. Preplay consequently yields symmetric duopoly, in which the firms have identical capital stocks and goods prices are at the investment boundary. The Stackelberg intuition that a "leader" can profitably foreclose the market through preemptive investment only holds under the extreme, unrealistic "static market" assumptions of no demand growth, no demand uncertainty, and non-depreciating capital.

6 Conclusion

I derive subgame-perfect equilibria of a dynamic capital accumulation game in which investment is irreversible and demand is stochastic, paying special attention to a class of strategies characterized by the small firm preempting, as cheaply as possible, the larger firm's preemptive investment behavior. The results have strong implications for the Stackelberg-like results found in Stackelberg-Spence-Dixit models. In particular, the intuition that a leader can profitably foreclose the market through preemptive investment depends on extreme, unrealistic static market assumptions of no demand growth, no demand uncertainty, and non-depreciating capital. This result highlights, more generally, the danger of applying intuition derived in static models to dynamic settings.

In our dynamic setting a natural Markov perfect equilibrium supports Cournot *behavior,* despite the fact that the Cournot *strategy* does not itself support a dynamically consistent equilibrium. This result, similar in spirit to that of Kreps and Scheinkman (1983), illustrates the robust nature of Cournot behavior. Ironically, it is precisely the commitment power of capital, which we generally associate with Stackelberg behavior, that drives this Cournot outcome. In a static setting the commitment power of capital allows the "leader" to credibly preempt the "follower's" investment; in a dynamic setting it allows the follower to credibly preempt the leader's preemptive investment.

The fact that a Markov perfect strategy of "preempting preemption" generates the same equilibrium behavior as the open-loop Cournot equilibrium is not a coincidence. If capital represents a strong commitment to production, then preemptive investment represents a credible threat. If firms can take frequent actions and possess a linear-incremental technology, then the cost of this preemptive behavior is low, because firms can invest "a little bit, just before" their opponent would otherwise invest. When preemptive investment is both cheap and credible, the only way to discourage it is to make it unprofitable. The preemptive preemption strategy represents a low cost mechanism for doing so. Consequently, while open-loop strategies are generally imperfect, in these environments, in which preemption is both cheap and credible, analysis of the open-loop Cournot strategy provides a convenient method for determining closed-loop equilibrium behavior.

A Appendix

A.1 Stochastic Cost of Capital

If the unit cost of new capital, c_t, is stochastic, then the firm's optimization problem consists of

$$V_i\left(k_{i,t},k_{-i,t},X_t,c_t\right) = \max_{\{I_{i,t+s}\}} \mathbf{E}_t\left[\int_0^\infty e^{-rs}\left(k_{i,t+s}\left(\frac{X_{t+s}}{k_{t+s}}\right)^\gamma ds - c_t dI_{i,t+s}\right)\Big|\{I_{-i,t+s}\}\right]. \quad (41)$$

This is homogeneous of degree one in both 1) k_1, k_2, X and c jointly, and 2) X^γ and c jointly, so a firm's investment strategy can be characterized as a market share dependent investment threshold in $p \equiv P/c$, the goods price to capital cost ratio. Our primary analysis is unchanged, except for our calculation of Π, the capitalized value, at the investment boundary, of capital that currently produces a unit flow of revenue.

Now suppose that the unit cost of capital follows a geometric Brownian processes with drift and volatility μ_c and σ_c, let $\mu_P = \gamma\left(\mu+\delta-(1-\gamma)\frac{\sigma^2}{2}\right)$ and $\sigma_P = \gamma\sigma$ again denote the drift and volatility in goods market prices absent the control, and ρ denote the correlation between the two processes. Let $p_t \equiv P_t/c_t$ denote the price/cost process and p^* denote the relative-price investment threshold, and define $\tau \equiv \min_{s\geq 0}\{p_s = p^*\}$. Finally, for notational convenience, for any process ζ let $\zeta_{t,s}$ denote ζ_t/ζ_s.

Using the fact that P is a geometric Brownian process up until τ, we have that

$$\begin{aligned}
\pi\left(P_0, p_{\tau,0}\right) &\equiv \mathbf{E}_0\left[\int_0^\infty e^{-rt}\left(e^{-\delta t}P_t\right)ds\big|p_{\tau,0}\right] \\
&= \mathbf{E}_0\left[\left(\int_0^\tau + \int_\tau^\infty\right)e^{-(r+\delta)t}P_t dt\big|p_{\tau,0}\right] \\
&= P_0\pi + \mathbf{E}_0\left[e^{-(r+\delta)\tau}P_\tau\big|p_{\tau,0}\right](\Pi-\pi)
\end{aligned} \quad (42)$$

where $\pi = \frac{1}{r+\delta-\mu_P}$. Letting $\alpha \equiv \frac{\text{cov}(dP/P,dp/p)}{\text{var}(dp/p)} = \frac{\sigma^2-\rho\sigma_P\sigma_c}{\sigma_P^2+\sigma_c^2-2\rho\sigma_P\sigma_c}$, then p_τ and P_τ/p_τ^α are uncorrelated, so

$$\begin{aligned}
\mathbf{E}_0\left[e^{-(r+\delta)\tau}P_\tau\big|p_{\tau,0}\right] &= P_0\mathbf{E}_0\left[e^{-(r+\delta)\tau}\mathbf{E}_0\left[P_{\tau,0}p_{\tau,0}^{-\alpha}\right]p_{\tau,0}^\alpha\big|p_{\tau,0}\right] \\
&= P_0 p_{\tau,0}^\alpha\mathbf{E}_0\left[e^{-(r+\delta-m)\tau}\right]
\end{aligned} \quad (43)$$

where

$$
\begin{aligned}
m &= \tfrac{1}{\tau}\ln \mathbf{E}_0\left[P_{\tau,0} z_{\tau,0}^{-\alpha}\right] \\
&= \tfrac{1}{\tau}\ln \mathbf{E}_0\left[P_{\tau,0}^{1-\alpha} c_{\tau,0}^{\alpha}\right] \\
&= (1-\alpha)\left(\mu_P - \tfrac{\alpha\sigma_P^2}{2}\right) + \alpha\left(\mu_c - \tfrac{(1-\alpha)\sigma_c^2}{2}\right) + \alpha(1-\alpha)\rho\sigma_P\sigma_c .
\end{aligned}
$$

Then

$$
\mathbf{E}_0\left[e^{-(r+\delta-m)\tau}\right] = \left(\tfrac{p_0}{p^*}\right)^{\theta} \tag{44}
$$

where θ is the positive root of

$$
\tfrac{\sigma_p^2}{2}x^2 + \left(\mu_p - \tfrac{\sigma_p^2}{2}\right)x - (r+\delta-m) = 0
$$

where μ_p and σ_p are the drift and volatility of the price/cost ratio,

$$
\begin{aligned}
\mu_p - \tfrac{\sigma_p^2}{2} &= \left(\mu_P - \tfrac{\sigma_P^2}{2}\right) - \left(\mu_c - \tfrac{\sigma_c^2}{2}\right) \\
\sigma_p^2 &= \sigma_P^2 + \sigma_c^2 - 2\rho\sigma_P\sigma_c .
\end{aligned}
$$

Combining equations (42), (43) and (44) yields

$$
\pi\left(P, p/p^*\right) = P\pi + P\left(\tfrac{p}{p^*}\right)^{\theta-\alpha}(\Pi - \pi). \tag{45}
$$

Finally, letting $\tilde{p}$ denote the uncontrolled price/cost ratio,

$$
\lim_{\epsilon\downarrow 0}\frac{d\pi\,(P, p/p^*)}{d\tilde{p}}\bigg|_{\tilde{p}=p^*-\epsilon} = \lim_{\epsilon\downarrow 0}\frac{d\pi\,(P,1)}{d\tilde{p}}\bigg|_{\tilde{p}=p^*+\epsilon}
$$

which, using the explicit functional form provided in equation (45), together with $\frac{dP}{dp}\big|_{p<p*} = \alpha$, $\frac{d(Pp^{-\alpha})}{dp}\big|_{p<p^*} = 0$ and $\frac{dp}{d\tilde{p}}\big|_{\tilde{p}=p^*} = \frac{\text{cov}(c,\tilde{p})}{\text{var}(\tilde{p})} = \alpha - 1$, implies

$$
\alpha\pi + \theta(\Pi - \pi) = (\alpha-1)\Pi,
$$

or

$$
\Pi = \left(\frac{\theta-\alpha}{\theta-\alpha+1}\right)\pi. \tag{46}
$$

A.2 Proof of Propositions

Proof of Proposition 4.1

Lemma A.1. *If firms always invest at some price threshold $P^*(s)$ to maintain market share, then the value of a firm at the price threshold, relative to the replacement cost of its capital stock, is*

$$Q^* \quad = \quad 1 + \left(\tfrac{P^*(s)}{P_c^*} - 1\right)\left(\tfrac{\beta}{\beta-1}\right). \tag{47}$$

Proof of lemma: Below the investment boundary

$$V_i^*(s,P) \quad = \quad k_i \pi P + \left(\tfrac{P}{P^*(s)}\right)^{\beta/\gamma}\left(V_i^*\left(s, P^*(s)\right) - k_i \pi P^*(s)\right), \tag{48}$$

so

$$\begin{aligned}
\left.\frac{dV_i^*(s,P)}{dX}\right|_{X=X_s^*-} &= \left.\frac{dP}{dX}\frac{dV_i^*(s,P)}{dP}\right|_{X=X_s^*-} \\
&= \tfrac{\gamma P^*(s)}{X_s^*}\left(k\pi + \tfrac{\beta/\gamma}{P^*(s)}\left(V_i^*\left(s,P^*(s)\right) - k\pi P^*(s)\right)\right) \\
&= \tfrac{\beta}{X_s^*}\left(V_i^*\left(s,P^*(s)\right) - \Pi P^*(s)\right)
\end{aligned} \tag{49}$$

where the last line follows from $\pi/\Pi = \frac{\beta}{\beta-\gamma}$. If the firm invests to maintain market share, then

$$\left.\frac{dk_i}{dX}\frac{d}{dk_i}\left(k_i Q_i - k_i\right)\right|_{X=X_s^*+} \quad = \quad \frac{k_i\left(Q^*-1\right)}{X_s^*}. \tag{50}$$

The previous two equations, together with the smooth pasting condition, imply the lemma.

Proof of the proposition: If only the small firm invests at the Cournot goods price threshold, then

$$\left.\tfrac{ds}{dX}\right|_{X=X_s^*+} \quad = \quad \left.\tfrac{ds}{dK}\tfrac{dK}{dX}\right|_{X=X_s^*+} \quad = \quad \left.\left(\tfrac{1-s}{K}\right)\tfrac{dK}{dX}\right|_{X=X_s^*+}, \tag{51}$$

and $\left.\frac{dP/dX}{P}\right|_{X=X_s^*+} = \left.\frac{dP^*/dX}{P^*}\right|_{X=X_s^*+}$ so

$$\frac{1}{X_s^*} - \left.\frac{\frac{dK}{dX}}{K}\right|_{X=X_s^*+} \quad = \quad \frac{\left.\frac{ds}{dX}\right|_{X=X_s^*+}}{1-\gamma s}, \tag{52}$$

which taken together imply

$$\left.\frac{ds}{dX}\right|_{X=X_s^*+} \quad = \quad \frac{1/X_s^*}{\frac{1}{1-s} + \frac{1}{1-\gamma s}}. \tag{53}$$

Substituting equations (49) and (53) into (13), and using $\Pi P_C^*(s) = \frac{1}{1-\gamma s}$, together with $\frac{dk}{ds}\big|_{X=X_s^*+}$ equals zero for the large firm and $\frac{k_2}{(1-s)^2}$ for the small, yields

$$\frac{dV_2^*\big(s, P_C^*(s)\big)}{ds} = \beta\left(\tfrac{1}{1-s} + \tfrac{1}{1-\gamma s}\right)\left(V_2^*\big(s, P_C^*(s)\big) - \tfrac{k_2}{1-\gamma s}\right) \tag{54}$$

$$\frac{dV_1^*\big(s, P_C^*(s)\big)}{ds} = \beta\left(\tfrac{1}{1-s} + \tfrac{1}{1-\gamma s}\right)\left(V_1^*\big(s, P_C^*(s)\big) - \tfrac{k_1}{1-\gamma s}\right) + \tfrac{k_2}{(1-s)^2}. \tag{55}$$

Dividing the previous two equations by k_2, then using the fact that $\frac{k_1}{k_2} = \frac{s}{1-s}$ and solving the resultant differential equations, results in formulas for $Q_C^2\left(s, P_C^*(s)\right) = \frac{V_2^*(s, P_C^*(s))}{k_2}$ and $Q_C^1\left(s, P_C^*(s)\right) = \left(\frac{1-s}{s}\right)\frac{V_1^*(s, P_C^*(s))}{k_2}$. Imposing the boundary conditions $Q_i^C\left(\frac{1}{2}, P_C^*\left(\frac{1}{2}\right)\right) = 1 + \left(\frac{\gamma/2}{1-\gamma/2}\right)\frac{\beta}{\beta-1}$, which follows directly from Lemma A.1, and simplifying the result yields the proposition.

Proof of Proposition 4.2

Proof of the proposition: The proposition follows directly from the proof of Proposition 4.1, subject to the boundary condition $Q_{1/2-}^* \equiv Q_i^{sm}\big(1/2, P_C^*(1/2)\big)$, the average value of capital when firms have equal capacities and goods prices are equal to the symmetric Cournot goods price investment threshold. Because neither firm will invest until goods prices reach the monopoly threshold, and letting $\tau = \min_{t>0}\{P_t = P_m^*\}$, this boundary condition is given by

$$\begin{aligned} Q_{1/2-}^* &= \mathbf{E}_0\left[\int_0^\tau e^{-(r+\delta)t}P_t dt + e^{-(r+\delta)\tau}Q_i^{sm}\big(1/2, P_m^*\big)\,\big|P_0 = P_C^*(1/2)\right] \\ &= \pi P_C^*(1/2) + \left(\tfrac{P_C^*(1/2)}{P_m^*}\right)^{\beta/\gamma}\left(Q_i^{sm}\big(1/2, P_m^*\big) - \pi P_m^*\right) \\ &= \left(\tfrac{1}{1-\gamma/2}\right)\left(\tfrac{\beta}{\beta-\gamma}\right) + \left(\tfrac{1-\gamma}{1-\gamma/2}\right)^{\beta/\gamma}\tfrac{\gamma}{(\beta-1)(\beta-\gamma)}. \end{aligned} \tag{56}$$

Proof of Proposition 4.3

Suppose the small firm has market share s, and goods prices are $P_p^*(s)$. If, in response to a positive demand shock, the firms add capacity in proportion to their current market shares, then they will add sufficient capacity to prevent goods prices from exceeding $P_p^*(s)$. If the large firm deviates from this capacity response by dk_2, however, then the small firm will respond by altering its investment by $dk_1 = \left(\frac{dk_1}{dk_2}\right)dk_2$. Goods prices will still end up at the investment threshold, so $\frac{dP_p^*}{dk_2}\big|_{P=P_p^*} = \frac{dP}{dk_2}\big|_{P=P_p^*}$, which using $\frac{ds}{dk_2}\frac{dP_p^*}{ds} = \frac{dK}{dk_2}\frac{dP}{dK}\big|_{P=P_p^*}$ together with $\frac{ds}{dk_2} = -\frac{1}{K}\left(s - (1-s)\frac{dk_1}{dk_2}\right)$ and $\frac{dK}{dk_2} = 1 + \frac{dk_1}{dk_2}$ implies that

$$\frac{dP_p^*}{ds} = \frac{-\gamma\left(1+\frac{dk_1}{dk_2}\right)P_p^*}{s-(1-s)\frac{dk_1}{dk_2}}. \tag{57}$$

Eliminating $\frac{dk_1}{dk_2}$ from the previous equation using equations (28) and simplifying the result using $P_c^* = \Pi^{-1}$ yields the description of the preemption curve provided in the proposition.

Proof of Proposition 4.4

Proof of the proposition: Above the critical market share s_{pp}^*, the valuation of the big firm follows directly from Lemma (A.1) and the fact that the firm is indifferent to investment at the small firm's investment price threshold. The valuation of the small firm follows directly from equation (34), and the facts that $\frac{dV_i^*}{dX}\big|_{X=X_s^*-} = k_i\beta\left(Q_i^{pp}\left(s, P_{pp}^*(s)\right) - \frac{P_{pp}^*(s)}{P_c^*}\right)$ and $\frac{dk_1}{ds}\big|_{X=X_s^*+} = \frac{k_2}{(1-s)^2}$, which imply

$$\frac{dV_1^*}{ds} \quad = \quad \frac{\gamma\beta}{1-s}\left(\frac{P_{pp}^*(s)}{P_c^*-(1-\gamma)P_{pp}^*(s)}\right)\left(V_1^*(s) - k_1\frac{P_{pp}^*(s)}{P_c^*}\right) + \frac{k_2}{(1-s)^2}. \tag{58}$$

Dividing the previous equation by k_2 and letting $v(s) \equiv \frac{V_1^*(s)}{k_2}$ yields the characterization of the small firm's value for $s > s_{pp}^*$ provided in the proposition.

Below s_{pp}^* the valuation of both firms follows directly from the proof of Proposition 4.1 and the fact that the small firm invests at the Cournot investment threshold $P_C^*(s)$.

A.3 Shared Monopoly Parameter Restriction

The large firm never has an incentive to deviate from the shared monopoly strategy provided it is always worth at least as much ceding market share to the smaller firm as it is investing to maintain market share. This is the case if and only if $\beta < \beta^*(\gamma)$ where $\beta^*(\gamma)$ is defined as the solution in β to

$$\min_{s\in(0,1/2)}\left\{Q_2^*(s, \tfrac{1}{2}, Q_{sm}^*) - \left(1 + \left(\frac{\gamma s}{1-\gamma s}\right)\frac{\beta}{\beta-1}\right)\right\} \quad = \quad 0 \tag{59}$$

where Q_2^* and Q_{sm}^* are as defined in equations (15) and (56).

Note that if $\beta = \beta^*(\gamma)$, and letting $s^*(\gamma) \equiv \arg\min\left\{Q_2^*\left(s, \frac{1}{2}, Q_{sm}^*\right) - \left(1 + \left(\frac{\gamma s}{1-\gamma s}\right)\frac{\beta}{\beta-1}\right)\right\}$, then $\frac{\partial}{\partial s}Q_2^*(s, \frac{1}{2}, Q_{sm}^*)\big|_{s=s^*} = \frac{\partial}{\partial s}\left(1 + \left(\frac{\gamma s}{1+\gamma s}\right)\frac{\beta}{\beta-1}\right)\big|_{s=s^*}$. The left hand side can be replaced using $\frac{\partial}{\partial s}Q_2^*(s, \frac{1}{2}, Q_{sm}^*) = \beta\left(\frac{1}{1-s} + \frac{1}{1-\gamma s}\right)\left(Q_2^*(s, \frac{1}{2}, Q_{sm}^*) - \frac{1}{1-\gamma s}\right)$ (see equation (54)), which implies that $s^*(\gamma) = \frac{3+\sqrt{5-4\gamma}}{2(1+\gamma)}$, and $\beta^*(\gamma)$ can thus be calculated as the solution in β to

$$Q_2^*\left(\frac{3-\sqrt{5-4\gamma}}{2(1+\gamma)}, \frac{1}{2}, Q_{sm}^*\right) \quad = \quad 1 + \left(\frac{3-\sqrt{5-4\gamma}}{2/\gamma-1+\sqrt{5-4\gamma}}\right)\frac{\beta}{\beta-1}.$$

Solving this numerically yields $1/\beta^*(\gamma) \approx 0.1747(1-\gamma)$. This approximate restriction is slightly too restrictive if $\gamma \lesssim 0.55$ or $\gamma \gtrsim 0.91$ (*i.e.,* in these cases $1/\beta$ can be slightly smaller than

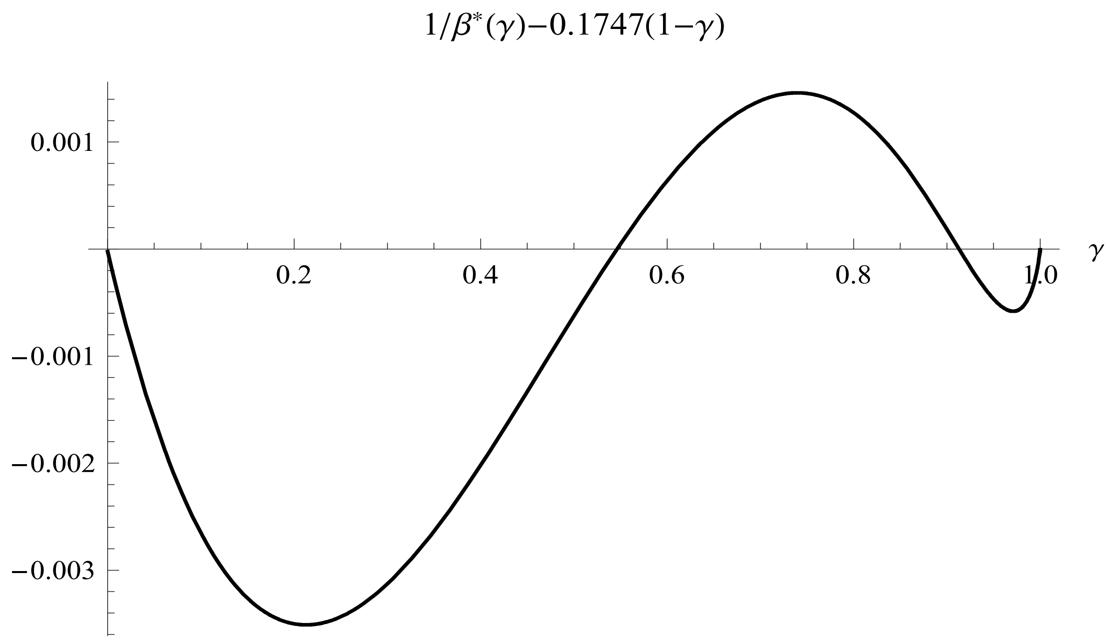

Figure 7: Shared Monopoly Parameter Restriction Refinement

The figure depicts deviations of $1/\beta^*(\gamma)$ from the linear approximation $0.1747(1-\gamma)$, where $\beta^*(\gamma)$ is the critical value of β in the parameter restriction employed in the derivation of the shared monopoly equilibrium (section 4.2).

$0.1747(1-\gamma)$), and not quite restrictive enough otherwise. Deviations of $1/\beta^*(\gamma)$ from $0.1747(1-\gamma)$ are depicted in Figure 7.

If β exceeds $\beta^*(\gamma)$, *i.e.,* if demand grows sufficiently slowly relative to the discount rate, then the large firm would sometimes find it profitable to preempt the investment of a small firm playing the simple shared monopoly strategy. For some positive $s < 1/2$ there are "sufficient" rents to capital, and the present value of future monopoly rents associated with playing the strategy are "small enough" because they come far in the future, that the large firm finds it unprofitable to cede market share to a small firm.

In these cases support of the shared monopoly outcome requires that the small firm preempts the large firm's preemptive investment. Given $\beta > \beta^*(\gamma)$, let $s^*(\gamma)$ be the larger positive solution in s to $Q_2^*(s, \frac{1}{2}, Q_{sm}^*) = 1 + \left(\frac{\gamma s}{1-\gamma s}\right)\frac{\beta}{\beta-1}$. For $s > s^*(\gamma)$ the large firm will not preempt the small firm's investment, because the large firm's value is greater playing the shared monopoly strategy. Just below $s^*(\gamma)$, however, the large firm would find it more profitable, if the small firm played the simple shared monopoly strategy, to maintain its market share by preempting some of the small firm's investment. In order to discourage this preemptive investment, the small firm must invest more aggressively, at the shared monopoly preemption price threshold. This threshold is defined implicitly by an equation analogous to equation (33): for each s $P_{smp}^*(s)$ solves

$$\frac{(1-s)\left(P_{smp}^*(s)-P_c^*\right)}{P_{smp}^*(s)^{1/\gamma}} = \frac{(1-s^*(\gamma))\left(P_C^*(s^*(\gamma))-P_c^*\right)}{P_C^*(s^*(\gamma))^{1/\gamma}}. \tag{60}$$

The shared monopoly preemptive preemption strategy consists of the small firm investing at the shared monopoly preemptive preemption threshold, $P_{smpp}^*(s) = \min\{P_C^*(s), P_{smp}^*(s)\}$ if $s < 1/2$ and $P_{smpp}^*(1/2) = P_m^*$. Firms may be valued explicitly using the techniques employed in section 4.

A.4 Partial Preemptive Preemption Strategies

When the boundary condition is below the symmetric duopoly investment threshold but above the competitive investment price threshold, $P_p^*(1/2) \in \left(P_c^*, P_C^*(1/2)\right)$, then the preemption curve describes strategies that involve "partial preemption" at $s = 1/2$. These correspond to $\frac{dk_1}{dk_2}\big|_{s=1/2} \in (-1, 0)$. If firms have equal market shares, and one firm deviates by adding extra capacity, the other firm reduces the capacity it adds, but by less than the full deviation. These strategies represent "passive" punishment strategies. If the large firm deviates by adding capacity and capturing market share, it lowers the price level at which the small firm invests, but by less than it lowers prices in the goods market. The cost imposed on the large firm by the reduced investment price threshold is sufficient to discourage deviations, but unlike the tit-for-tat punishment strategy considered in the previous section, the "punishment" comes in the future, when prices reach the new lower investment price threshold.

When the boundary condition is at the competitive investment price threshold, $P_p^*(1/2) = P_c^*$, then $\frac{dP_p^*}{ds} = 0$, and consequently $P_{pp}^*(s) = P_c^*$, for all s. This corresponds to the case of "full preemption," when a firm's investment always reduces its opponent's future investment by an equal amount, $\frac{dk_i}{dk_{-i}}\big|_s = -1$.

The curves corresponding to $P_p^*(1/2) > P_C^*(1/2)$ specify strategies that fail to satisfy the Markov restriction. These strategies attempt to support goods market prices that exceed the Cournot investment price threshold, supported by "active" punishment ($\frac{dk_i}{dk_{-i}}\big|_{s=1/2} > 0$, so investment by the large firm elicits immediate investment by the small firm) that is less aggressive than the tit-for-tat punishment that supports the shared monopoly outcome. These strategies require, however, that the small firm delays investment until prices strictly exceed the Cournot investment threshold, despite the fact that earlier investment by the small firm would not elicit a punitive response. When bygones are bygones the small firm cannot commit itself to delaying this investment, and these strategies are therefore not Markov.

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