

# Real Estate and the Economy: Aggregate Implications of Irreversible Investment\*

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## Abstract

Irreversibility prevents the continual allocation of capital to its most productive use, with first-order economic consequences. This paper analyzes the aggregate implications, introducing a general equilibrium model with two consumption goods and irreversible investment, and showing that developers' optimal behavior endogenously generates a business cycle: periods of more intensive real estate development are associated with greater consumption growth, even when fundamental shocks are stationary. I also consider an array of business cycle-dependent macroeconomic implications, including the impact of irreversibility on the term structures of interest rates and interest rate volatility, consumption risk premia, and forward prices and forward price volatilities.

*Keywords:* Investment; Irreversibility; Real Estate; Tobin's  $Q$ ; Real Options; General Equilibrium; Multiple Goods; Interest Rates; Asset Pricing.

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# 1 Introduction

Residents of declining cities consume “too much” housing services. The current population of Detroit, given a clean slate on which to build and resources sufficient to duplicate the current scale of the city, would not choose to do so.

The same investment irreversibility that locks households into sub-optimally high levels of housing consumption in bad times leads to “too little” investment in new housing, and consequently too little housing consumption, in good times. Developers hedge against possible future regret— which can occur because drops in demand cannot be offset by drops in supply— by building less than they would if investment were reversible. As a result, households and businesses consume too little housing in areas experiencing growth, in the sense that the marginal impact of new development on felicity (*i.e.*, the relative price of housing services) exceeds the marginal impact on felicity from alternative uses of capital.

Large sectors of the economy are characterized by similar frictions, which prevent the continual allocation of capital to its most productive use. While real estate is an obvious, important, example of an industry characterized by irreversible investment and fixed adjustment costs, most investment is subject to these frictions. Semiconductor manufacturers spend billions on the facilities that will fabricate their next generation of chips, capital that has essentially no alternative use. Cellular phone service providers incur huge fixed costs developing the network for their next generation of service, investment that renders their existing network obsolete and is itself destined for obsolescence.

These frictions differ fundamentally from those considered by Grossman and Laroque (1990), who examine how transaction costs at the consumer level, resulting from the absence of a rental market, affect individuals’ consumption of durables, while implicitly allowing for frictionless adjustments in the aggregate stock of durables. In order to highlight the impact of investment frictions that prevent costless adjustment in the stock of durables, we will assume a perfect rental market exists for the service flow provided by durables, allowing individuals to frictionlessly adjust the level of consumption of these services.

These investment frictions, by preventing the continual allocation of capital to its most productive use, have important, but largely ignored, economic consequences. The unavoidable misallocation of capital caused by these frictions leads to a sub-optimal composition of consumption, with marginal felicity from investment in new productive capacity differing across goods. This misallocation affects both the marginal rate of substitution across time and the riskiness of future

consumption, and thus both interest rates and equity premia.

Piazzesi, Schneider and Tuzel (2005) (hereafter “PST”) also consider how the composition of housing and non-housing consumption affects the marginal rate of substitution when utility is non-separable, and show that housing’s expenditure share forecasts excess returns on stocks.<sup>1</sup> They focus on the consumption side, however, taking the supply of housing as exogenous. In this paper supply is endogenous, and consistent with developers maximizing property value subject to constraints imposed by construction technology.<sup>2</sup> We focus on the restrictions that irreversibility and adjustment costs place on the equilibrium evolution of supply, and how these restrictions, which limit aggregate adjustments in the composition of consumption, impact interest rates, risk premia and forward prices.

The impact of the misallocation of capital that results from investment constraints depends fundamentally on the business cycle, because the underlying friction, irreversibility, is asymmetric. Investment constraints bind in recessions, when irreversibility prevents the reallocation of capital to more productive uses, but not in expansions, when investment naturally occurs in the sectors characterized by irreversibility. Consequently, the impact of sub-optimal capital allocation is greater in weak economic environments and at shorter horizons. Conversely, in strong economic environments, and at longer horizons, the relatively high supply elasticity of the goods produced by the sectors characterized by irreversibility mitigate the impact of these investment frictions.

In order to study the aggregate economic effects of irreversibility in greater detail, we introduce a preference based, general equilibrium model with two consumption goods and irreversible investment. We explicitly characterize the optimal investment plan of competitive heterogeneous agents. We then generate analytic expressions for the term structures of interest rates and interest rate volatilities, consumption risk premia, and forward prices and forward price volatilities, which facilitates our study of business cycle implications. Note these results are qualitative in nature, and *not* intended to address the equity premium puzzle. Our model employs standard constant relative risk aversion (CRRA) preferences over the aggregate consumption bundle. Generating a significant equity premium with these preferences requires highly volatile aggregate consumption, and

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<sup>1</sup> Yogo (2006) also considers the impact on asset returns of nonseparable utility over durable and nondurable consumption, but with a definition of durables that excludes real estate.

<sup>2</sup> Mamaysky (2001) also considers endogenous durable goods production when consumers have nonseparable preferences. The investment technology he considers, however, is linear, incremental and free of adjustment costs, and is consequently ill-suited to studying real estate development. His results, which are qualitatively different from those presented here, are driven by a counter-cyclical rate of nondurable consumption.

consequently implausibly high volatility in some significant component (or components) of consumption.<sup>3</sup>

These results will also depend implicitly on the choice of numeraire, because the inter-temporal rate of substitution for one good depends, when using an alternative numeraire, on both the inter-temporal rate of substitution and relative prices. That is, the “risk-free” interest rate with respect to one numeraire involves price speculation with respect to any other.

Rather than using any particular consumption good as numeraire, we will use appropriately aggregated consumption. While this choice is relatively standard in the Macro literature, it represents a departure from the (relatively limited) literature on asset pricing in multi-good economies, which typically chooses one consumption good arbitrarily as numeraire.<sup>4</sup> Using aggregated consumption has three great advantages over using an arbitrary consumption good: it yields results that are directly welfare relevant, makes the role of relative price changes clear, and is consistent with standard practice in the real world.<sup>5</sup>

Irreversible investment tends to occur in expansions, *i.e.*, in periods in which output grows faster than its unconditional mean. Developers are more likely to build new buildings when the non-housing sector grows faster than average. But this investment also *causes* consumption to grow faster: when the investment constraint does not bind we see welfare improving adjustments in the composition of consumption. These have a direct, positive impact on GDP. That is, nonseparable preferences and the irreversibility constraint together generate time-series variation in the sensitivity of real consumption growth to fundamental shocks, with consumption more exposed to fundamentals when the irreversibility constraint does not bind.

Because consumption grows faster when the irreversibility constraint does not bind and investment induces welfare improving adjustments in the composition of consumption, expansionary economies are associated with a quickly declining marginal rate of substitution, and consequently higher real interest rates. The short and long ends of the term structure have different sensitivities to economic conditions, however, resulting in a term structure slope that is hump-shaped in economic

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<sup>3</sup> PST generate a significant equity premia using similar preferences and data on the housing consumption share. There calibration, however, implies implausible variation in the level of housing consumption. This is discussed in greater detail in appendix A.

<sup>4</sup> See, for example, Richard and Sundaresan (1981), Sundaresan (1984), Mamaysky (2001), Kogan (2001), and Gomes, Kogan and Yogo (2006).

<sup>5</sup> Results in this numeraire are consistent with those we would get if we included money explicitly as the medium of transaction for consumption goods, and adjusted nominal dollar results for inflation by deflating price with a chained Fisher index, consistent with the methodology employed by the U.S. Department of Commerce’s Bureau of Economic Analysis (BEA) when compiling the National Income and Product Accounts (NIPA).

strength. The model predicts a term structure that is low and flat in deep recessions, low and upward sloping in mild recessions, high and flat in average economic conditions, and high and downward sloping in expansions. The model also predicts a volatility term structure that is unconditionally downward sloping, but conditionally hump-shaped, with a peak that is higher and closer to the short end in stronger economies. Together these imply that the slopes of the term structures of interest rates and interest rate volatilities correlate positively.

Irreversibility and adjustment costs also result in pro-cyclical consumption risk premia. Adjustments in the composition of consumption amplify the impact of fundamental shocks on aggregate consumption, but these adjustments only occur in expansions, making consumption more variable in strong economic environments and at longer horizons, when adjustments are more likely. Consequently, the term structure of consumption risk premia slopes upward, and more steeply so in expansions than in recessions.

Forward prices, relative to spot prices, are pro-cyclical for the non-housing good, but counter-cyclical for housing services. Absent the development of new buildings, because of both natural growth in the non-housing sector and depreciation in the housing sector, the price of non-housing goods tends to fall, and the price of housing services tends to rise. That is, in recessions, when the investment constraint binds and non-housing goods are becoming relatively plentiful while housing services become increasingly scarce, forward markets for the non-housing good are backwarddated (downward sloping, with the price for future delivery lower than the spot), while forward markets for housing services are in contango (upward sloping, with the price for future delivery higher than the spot). In expansions, when non-housing capital is diverted to the housing sector making non-housing capital relatively scarce and housing capital relatively plentiful, the situation is reversed, with non-housing good forward markets in contango and housing services forward markets backwarddated. The Samuelson hypothesis holds for both consumption goods, with forward price volatility decreasing with time-to-delivery. Growth in the supply of non-housing capital, which lowers the price of the non-housing good, leads to a diversion of resources into the housing sector, raising the price of the non-housing good, partially off-setting the price impact of natural growth. Growth in the supply of non-housing capital also raises the price of housing services, but diversion of resources into the housing sector lowers the price of housing services, again off-setting the price impact of natural growth. We also see more forward price volatility in recessions, because supply is less elastic. Because strong economic conditions result in steeply downward sloping forward

price volatilities, the slopes of the term structures of forward prices and forward price volatilities are negatively correlated for non-housing goods and positively correlated for housing services.

A significant, pro-cyclical component of the aggregate value of real estate is due to growth options, *i.e.*, the right to redevelop on a larger scale in the future. This component of value exhibits a high degree of cross sectional variation. For our default parameterization, the contribution of real options to overall value ranges from under two percent for large buildings to more than 40 percent for small buildings, and accounts for eight to eleven percent of value in the aggregate.

Real options contribute to a building's value without contributing to its current revenues, so smaller buildings, for which real options represent a more significant component of value, have lower capitalization rates than large buildings. Because real options' contribution to buildings' values is pro-cyclical, the cap-rate spread between large and small buildings is also pro-cyclical. Because development of new buildings tends to occur in expansions, this implies that cap-rate spreads between new and old buildings predict new development: new development is more likely to occur when the cap-rate spread is high.

Consistent with observed investment patterns in real estate markets, and more generally the micro-evidence on plant-level investment, investment at the project level is lumpy, characterized by periods of intense activity between which no investment occurs. The model allows for an explicit characterization of the expected time between capacity adjustments, and the dependence of this timing on the magnitude of the adjustment costs associated with investment, providing additional cross-sectional implications.

Finally, the analysis predicts a "natural" degree of heterogeneity, *i.e.*, a specific distribution of the density of development. In particular, the model predicts that the ratio of the sizes of the tallest and shortest buildings in the city will depend in a well defined way on the economic primitives, and that there will be fewer tall buildings, with the number of buildings of a given size inversely related to size. The degree of heterogeneity is, however, less than is socially optimal. Competition leads developers to incur the adjustment costs associated with redevelopment (loss of capital-in-place) "too soon," which leads to development on a scale that is too small to efficiently utilize the resources employed, resulting in a dead weight loss.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 motivates the strategy developers employ in equilibrium heuristically, and section 4 demonstrate the equilibrium formally. Section 5 considers properties of the equilibrium, including the expect-

tation and variance of time between capacity adjustments, and the equilibrium level of economic heterogeneity, and how this heterogeneity differs from the socially optimal level. Section 6 derives the term structures of expected consumption growth and expected consumption growth volatility, and how these vary over the business cycle. Sections 7 and 8 consider asset pricing implications, deriving the term structures of interest rates and consumption risk premia, and the expected returns on real assets, respectively. Section 9 concludes.

## 2 The Model

We will think informally of the model, an extended “Lucas-tree” economy, as consisting of apple trees and buildings. Apple trees grow in the “country” and provide a flow of a consumption good, apples. They may also be used to build buildings in the “city,” which provide a service flow, housing.

Formally, we have an economy in which there are two non-storable consumption goods, “non-housing goods and services” and “housing services.” These goods are produced by firms and consumed by households.

A continuum of competitive, value maximizing firms owns all the means of production in the economy, non-housing capital and “buildings,” where a building consists of a quantity of housing capital together with the site on which this housing capital is located. These sites exist in fixed supply. Ownership of the means of production is widely dispersed among firms, so each firm’s holdings of each type of capital are small (*i.e.*, infinitesimal) relative to the corresponding aggregate.

Firms sell their output in a competitive goods market, through which households purchase the goods and services they consume. Firms are owned by, and pay dividends to, households. Ownership is widely dispersed among households, so each household’s indirect holdings of each type of capital are small relative to the corresponding aggregate.

### 2.1 Production and Capital

Flows of the two goods are costlessly produced in proportion to the quantity of the corresponding capital stocks employed in their productions. Because production is proportional to the levels of the stocks, we will use  $Y_t$  and  $Q_t$  to denote both the aggregate time- $t$  flows of non-housing goods and housing services, respectively, and the aggregate time- $t$  levels of the corresponding stocks.

### 2.1.1 Non-Housing Capital

The non-housing capital stock, in addition to producing a flow of the non-housing good, grows stochastically over time according to a technology with constant returns to scale, and may also be converted irreversibly into housing capital, *i.e.*, apple trees grow, and may be used to construct buildings. The instantaneous evolution of the aggregate non-housing capital stock is therefore given by

$$dY_t = Y_t \left( \frac{dX_t}{X_t} \right) - dI_t \quad (1)$$

where  $X_t$  is the natural growth process for non-housing capital and  $dI_t$  is the quantity of non-housing capital diverted to the housing sector over the interval. The natural growth process  $X_t$  evolves according to

$$dX_t = X_t \mu_X dt + X_t \sigma_X dB_t \quad (2)$$

where  $\mu_X$  is the average growth rate for non-housing capital in the absence of conversion to housing capital,  $\sigma_X$  is the volatility of the growth, and  $dB_t$  is an increment to a standard Wiener process. Cumulative real investment in housing up to time- $t$ ,  $I_t$ , is a non-negative, non-decreasing singular process.

### 2.1.2 Housing Capital

Housing capital exists at sites with a measure normalized to one. This housing capital naturally depreciates at a constant rate  $\delta$ , and the aggregate stock increases only when firms that own sites, whom we will refer to as “developers,” convert non-housing capital irreversibly into housing capital. That is, the “city” grows in response to development, and buildings (or the quality of the service flow they provide) depreciate over time. The instantaneous evolution of the aggregate housing capital stock is therefore given by

$$dQ_t = -Q_t \delta dt + \frac{dI_t}{NCR_t} \quad (3)$$

where  $\delta$  is the depreciation rate,  $dI_t$  is the quantity of non-housing capital employed developing housing capital over the interval, and  $NCR_t$  is the time- $t$  “net unit conversion rate.” This net unit



conversion rate will depend, due to the nature of the conversion technology, on both 1) the timing of the capacity adjustment, and 2) on which developers are building and at what scale. The conversion rate will thus be determined endogenously as part of the equilibrium strategy.

The conversion technology available to developers entails adjustment costs, has decreasing returns-to-scale, and varies over time. Specifically, the “cost,” in non-housing capital, of adjusting the capacity at a site from  $q_{\text{old}}$  to  $q_{\text{new}}$  at time  $t$  is

$$c_t K(q_{\text{old}}, q_{\text{new}}) \quad (4)$$

where  $c_t$  is the inverse productivity of vintage- $t$  housing capital (or time- $t$  “conversion rate”) and  $K(\cdot, \cdot)$  is the adjustment cost function. The inverse productivity of vintage- $t$  housing capital evolves according to

$$dc_t = c_t \mu_c dt + c_t \sigma_c dB'_t \quad (5)$$

where  $\mathbf{E}[dB_t dB'_t] = \rho dt$ .<sup>6</sup> The adjustment cost function is homogenous degree- $\phi > 1$  in the pre- and post-development capacities jointly. It includes both “fixed” and “variable” costs of adjusting, which are respectively independent of, and dependent on, the magnitude of the adjustment. The variable cost is assumed to be increasing in the magnitude of the adjustment, and sufficiently convex to guarantee a unique solution to the firm’s problem. Taken together these assumptions imply  $K(x, y) = x^\phi K(y/x)$  where  $K(x) \equiv K(1, x)$  (homogenous degree- $\phi$ ),  $K(1) > 0$  (fixed adjustment costs),  $K'(x) > 0$  (costs increase in the magnitude of the adjustment) and  $\left(K(x)^{1/\phi}\right)'' \geq 0$  (guarantees uniqueness).

When we make explicit calculations that depend on the specification of  $K(\cdot)$ , we will assume fixed adjustment costs that are proportional to the existing capacity and variable (direct) costs that are Cobb-Douglas in the “deficit” between the desired capacity and the existing capacity less the loss from adjustment. That is, we will assume that the cost of investing is  $(q_{\text{new}} - (1 - \Delta)q_{\text{old}})^\phi$  where  $\Delta$  parameterizes the magnitude of the fixed adjustment costs and  $\phi$  parameterizes the cost-

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<sup>6</sup> Including the stochastic conversion rate is trivial in a risk-neutral framework in which the price of housing services follows an exogenously specified time-homogenous diffusion process, as a firm’s problem is then linear-homogenous in costs and prices jointly. In this case only a single source of uncertainty, the ratio of the two processes, is relevant for the firm’s investment strategy. Both sources of uncertainty remain relevant in the general equilibrium framework, however, because the pricing kernel, which is endogenous and non-trivial, depends directly on the evolution of the non-housing capital stock but only indirectly, through firms’ equilibrium investment strategy, on the conversion rate.

to-scale of development, in which case  $K(x) = (x - 1 + \Delta)^\phi$ . We will pay particular attention to the parameterization  $\Delta = 1$ , in which case  $K(q_{\text{old}}, q_{\text{new}}) = q_{\text{new}}^\phi$ , independent of  $q_{\text{old}}$ . This specification is common in the real estate literature, or more generally in the literature on development of real assets with capacity choice. It corresponds to the case when development entails abandonment of the existing capital at the site, *e.g.*, when a developer razes an existing building so that she can build a new, larger building in its place.

## 2.2 Preferences

Households' preferences admit a representative consumer with standard CRRA preferences over aggregate consumption  $C_t = Y_t^a Q_t^{1-a}$  for some constant  $a \in (0, 1)$ ,

$$\mathbf{E}_t \left[ \int_0^\infty e^{-\lambda s} u(C_{t+s}) ds \right] \quad (6)$$

where  $u(C_t) = C_t^{1-\gamma_C} / (1 - \gamma_C)$ , and the rate of time preference  $\lambda$  and the coefficient of CRRA  $\gamma_C \geq 0$  are both constant.<sup>7</sup> The Cobb-Douglas consumption aggregator implies a unitary elasticity of intra-temporal elasticity. While restrictive, this assumption is roughly consistent with the data, which shows very little variation in the expenditure shares on housing (see, for example, PST). We will generally restrict attention to the case when the elasticity of inter-temporal substitution exceeds the elasticity of intra-temporal substitution, *i.e.*, to the case when  $\gamma_C > 1$ .<sup>8</sup>

## 3 Motivating the Equilibrium Analysis

Before formally demonstrating an equilibrium of this economy, we will heuristically motivate the general form of the investment strategy that developers employ in equilibrium, and consider some general properties of the evolution of aggregate variables that result from this behavior. We will formalize the equilibrium argument in section 4.

Demonstrating the formal equilibrium will be complicated by the fact that competitive agents' behavior is inconsistent with the solution to the social planner's problem. Real estate development entails large opportunity costs. Developing a building requires forgoing any ongoing assets that

<sup>7</sup> Of course when  $\gamma_C = 1$  equation (6) should use the log formulation, and utility is separable over the two goods.

<sup>8</sup> While non-housing goods and housing services are neither Hicksian compliments nor substitutes, because the intra-temporal elasticity of substitution is exactly one, if  $\gamma_C > 1$  then  $u_{YQ} < 0$  and marginal utility with respect to one good is decreasing in the level of the other.

exist at the site, *i.e.*, in order to redevelop a lot, the developer must tear down whatever already exists there. The loss of assets-in-place constitutes a fixed adjustment cost, and prevents incremental adjustments to the capital stock at any given address. These fixed adjustment costs mean developers' production sets are nonconvex, so the second welfare theorem does not apply. We cannot, therefore, solve the planner's problem to find the competitive equilibrium, and will be forced to consider the investment problem from the points of view of individual agents in the economy.

In considering firms' equilibrium investment behavior it proves convenient to use the non-housing consumption good, not aggregate consumption, as numeraire. Calculations using this numeraire are facilitated by employing an alternative representation for instantaneous utility, given by

$$u(Y_t, Q_t) = \text{sign}(1 - \gamma_C) \left( \frac{Y_t^{1-\gamma_Y}}{1 - \gamma_Y} \right) \left( \frac{Q_t^{1-\gamma_Q}}{1 - \gamma_Q} \right) \quad (7)$$

where  $\gamma_y \equiv 1 - a(1 - \gamma_C)$  and  $\gamma_q \equiv 1 - (1 - a)(1 - \gamma_C)$  are the representative consumer's risk aversion parameters for each of the consumption goods individually, and the constraints on  $a$  and  $\gamma_C$  imply that  $\gamma_y, \gamma_q, (1 - \gamma_y)(1 - \gamma_q)$  and  $\gamma_y + \gamma_q - 1$  are all non-negative.

### 3.1 The Developer's Problem

Developers invest to maximize value, *i.e.*, to maximize the expected discounted cash flows net of construction costs, where both the discount factor and the price of a project's output (housing services) are determined by the representative consumer's preferences and the evolution of the consumption processes, and consequently depend on the investment strategies other developers employ in equilibrium. Because investment entails a discrete adjustment cost, any given developer's investment plan will consist of discrete times  $t_i$  at which she develops to capacities  $q_i$  for  $i = 1, 2, \dots$ . The value of a site with capacity  $q_t$ , when the current aggregate production of the two goods are  $Y_t$  and  $Q_t$  and the conversion rate is  $c_t$ , is therefore

$$V(q_t, Y_t, Q_t, c_t) = \max_{\{t_i, q_i\}} \mathbf{E}_t \left[ \int_0^\infty e^{-\lambda s} \xi_{t,s}^Y (\tilde{q}_{t+s} P_{t+s} ds - dC_{t+s}) \Big| Y_t, Q_t, c_t \right] \quad (8)$$

where for notational convenience we let  $Z_{t,s} \equiv Z_{t+s}/Z_t$  for any process  $Z_t$ , and

- $e^{-\lambda s} \xi_{t,s}^Y$  is the representative consumer's marginal rate of substitution between  $t$  and  $t + s$  for

the numeraire (non-housing) consumption good, and

$$\xi_{t,s}^Y = \frac{u_Y(Y_{t+s}, Q_{t+s})}{u_Y(Y_t, Q_t)} = Y_{t,s}^{-\gamma_Y} Q_{t,s}^{1-\gamma_Q}, \quad (9)$$

- $P_{t+s}$  is the time- $(t + s)$  price of housing services, given by

$$P_{t+s} = \frac{u_Q(Y_{t+s}, Q_{t+s})}{u_Y(Y_{t+s}, Q_{t+s})} = \left( \frac{1-a}{a} \right) \frac{Y_{t+s}}{Q_{t+s}}, \quad (10)$$

which depends on other developers' investment strategies through its dependence on the aggregate state variables,

- $\tilde{q}_{t+s}$  and  $dC_{t+s}$ , housing capital at the site and the cost of investment, respectively, both depend on the firm's investment strategy and are given by

$$\tilde{q}_{t+s} = \begin{cases} e^{-\delta s} q_t & \text{if } t+s \leq t_1 \\ e^{-\delta(t+s-t_i)} q_i & \text{for } t_i < t+s \leq t_{i+1} \end{cases}$$

$$dC_{t+s} = \begin{cases} c_{t_i} \tilde{q}_{t_i}^\phi K(q_i / \tilde{q}_{t_i}) \Pi_{t_i}^I & \text{if } t+s = t_i \text{ for } i = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

where  $\Pi_t^I$  is the unit price of non-housing capital at time- $t$  (*i.e.*, the time- $t$  apple price of an apple tree that provides a unit flow of apples), and is given by

$$\Pi_t^I = \mathbf{E}_t \left[ \int_0^\infty e^{-\lambda s} \xi_{t,s}^Y X_{t,s} ds \right]. \quad (11)$$

### 3.2 The General Form of the Investment Strategy

Because each firm is individually small, and takes the aggregate processes as given, we can think of an individual firm's value function as depending on  $P_t$ ,  $c_t$  and  $z_t \equiv Y_t / (c_t Q_t^\phi)$ , which contain the same information as  $Y_t$ ,  $Q_t$  and  $c_t$ . That is, we can write the value of a firm with current capacity  $q_t$  as

$$V(q_t, P_t, c_t, z_t) = \max_{\{t_i, q_i\}} \mathbf{E}_t \left[ \int_0^\infty e^{-\lambda s} \xi_{t,s}^Y (\tilde{q}_{t+s} P_{t+s} ds - c_{t_i} \Pi_{t_i}^I \tilde{q}_{t_i}^\phi K(q_i / \tilde{q}_{t_i})) \Big| P_t, c_t, z_t \right]. \quad (12)$$

If  $z$  is Markovian and the evolution of the log-pricing kernel and the log-price of housing services only depend on  $z$ , then the finite dimensional distributions of  $\xi_{t,s}^Y$  and  $P_{t,s}$  are independent of  $\xi_t^Y$  and  $P_t$ , and the right hand side of the previous equation is homogenous degree- $\phi$  in  $q$  and  $P^{1/(\phi-1)}$  jointly, so

$$V(kq, k^{\phi-1}P, c, z) = k^\phi V(q, P, c, z). \quad (13)$$

The right hand side of equation (12) is also homogenous degree-1 in  $P$  and  $c$  jointly so we can rewrite the previous equation, after suppressing the dependence on  $z$  for notational convenience and letting  $v(q, p) \equiv V(q, P, c)/c$  where  $p \equiv P/c$ , as

$$v(q, p) = q^\phi v(1, q^{1-\phi} p). \quad (14)$$

This implies a multiplicative investment rule in  $p$ . A developer builds to some constant multiple  $\kappa$  (currently unknown) of its existing capacity  $q$  when the price of housing services scaled by the conversion rate reaches  $q^{\phi-1} p^*$ , where  $p^*$  (also currently unknown) denotes the price/conversion rate ratio at which a developer would optimally redevelop a site with unit capacity.

### 3.3 Evolution of the Aggregate Variables

Now suppose all developers follow a multiplicative investment strategy, with each redeveloping a site by some constant factor  $\kappa$  whenever  $p = P/c$  reaches some constant  $p^*$  times the  $\phi - 1^{th}$  power of the site's existing capacity. Suppose further that the initial distribution of projects' log-capacities is uniformly distributed between  $\ln q_t^{\min}$  and  $\ln q_t^{\min} + \ln \kappa$ , a distribution that is preserved cross-sectionally by the investment strategy. Under these assumptions the next developer to build, the one with the site currently developed at the lowest density, does so when

$$\frac{P_t}{c_t (q_t^{\min})^{\phi-1}} = p^*. \quad (15)$$

Given the assumed cross-sectional distribution of project capacities, the aggregate level of the housing capital stock is  $Q_t = \left(\frac{\kappa-1}{\ln \kappa}\right) q_t^{\min}$ , which together with  $P_t = \left(\frac{1-a}{a}\right) Y_t/Q_t$  implies some

firm builds, and aggregate capacity is consequently increasing, whenever

$$z_t \equiv \frac{Y_t}{c_t Q_t^\phi} = \left( \frac{1-a}{a} \right) \left( \frac{\ln \kappa}{\kappa - 1} \right)^{\phi-1} p^* \equiv z^*. \quad (16)$$

Because firms add capacity whenever  $z_t$  reaches  $z^*$ , we think of  $z_t$  as related to the business cycle. When  $z_t$  is close to  $z^*$ , near term expansions in the housing sector are likely, and we think of these as strong, or expansionary, economic conditions. When  $z_t$  is significantly below  $z^*$ , near term expansions in the housing sector are unlikely, and we think of these as weak, or recessionary, economic conditions. Later we will show that the expected rate of consumption growth is increasing in  $z_t$ , which justifies this interpretation.

This business cycle variable  $z_t$  evolves as a reflected geometric Brownian process. Below  $z^*$  it changes only in response to natural shocks to  $X_t$  and  $c_t$  and the natural depreciation of  $Q_t$ . At the critical threshold  $z^*$ , however, positive shocks to the non-housing capital stock / capital conversion cost ratio  $y_t \equiv Y_t/c_t$  elicit investment, resulting in both a decrease in  $Y_t$  and an increase in  $Q_t$ , which prevents  $z_t$  from ever exceeding  $z^*$ . Because  $z_t$  evolves as a reflected geometric Brownian process we can write it in terms of the uncontrolled process  $\zeta_t \equiv Y_0 X_{0,t}/c_t Q_t^\phi = z_0 e^{\phi \delta t} X_{0,t}/c_{0,t}$  as

$$z_t = z^* \left( \frac{\zeta_t}{M_t} \right) \quad (17)$$

where  $M_t = \max\{z^*, \max_{s < t} \{\zeta_s\}\}$ . On the interval  $(0, T \equiv \min\{s > 0 | \zeta_s = z^*\})$ , *i.e.*, up until some firm invests,  $M_t = z^*$ , so  $z_t$  and  $\zeta_t$  agree. After  $T$ ,  $M_t > z^*$ , and equation (16) says  $z_t/z^* = \zeta_t/M_t$ , *i.e.*, that the geometric distance from the controlled process to the reflecting barrier is the same as the geometric distance from the uncontrolled process to its maximum.

### 3.3.1 The Capital Stock Processes

At the barrier positive shocks elicit investment such that  $dz_t = 0$ , so

$$d \ln y_t = \phi d \ln Q_t. \quad (18)$$

The change in the housing capital stock in response to a positive shock at the investment bound-

ary may also be computed directly, and is given by

$$dQ_t = \frac{dI_t}{NCR_t} \quad (19)$$

where the numerator,  $dI_t$ , is the quantity of the non-housing capital converted into housing capital, and the denominator,  $NCR_t$  is the time- $t$  net unit conversion rate. The net unit conversion rate is the time- $t$  “cost” in non-housing capital of increasing the density of housing capital divided by the net increase in density,  $c_t (q_t^{\min})^\phi K(\kappa) / (\kappa q_t^{\min} - q_t^{\min})$ , or simplifying using  $Q_t = \left(\frac{\kappa-1}{\ln \kappa}\right) q_t^{\min}$ ,

$$NCR_t = \frac{c_t K(\kappa) (Q_t \ln \kappa)^{\phi-1}}{(\kappa-1)^\phi}. \quad (20)$$

We may also explicitly calculate  $dI_t$ , the numerator of equation (19), by noting that in the absence of the control the stock of non-housing capital evolves according to the natural growth process  $\ln X_t$ . Adding the non-housing capital used to increase the housing capital stock back to the first stock we have, therefore, that on the set on which development occurs

$$\frac{dY_t + dI_t}{Y_t} - \frac{dc_t}{c_t} = d \ln x_t \quad (21)$$

where  $x_t \equiv X_t/c_t$ .<sup>9</sup> Solving for  $dI_t$  yields

$$dI_t = Y_t(1 - \omega_t) d \ln x_t \quad (22)$$

where  $\omega_t \equiv d \ln y_t / d \ln x_t$ .

Rewriting equation (18) using equations (19), (20) and (22), and the fact that  $Y_t/c_t Q_t^\phi = z^*$  at the investment boundary, gives

$$\omega_t \ln x_t = \frac{\phi z^* (1 - \omega_t) \ln x_t}{\frac{K(\kappa)}{(\kappa-1)^\phi} (\ln \kappa)^{\phi-1}}, \quad (23)$$

---

<sup>9</sup> Equation (21) implicitly uses the fact that on the set on which development occurs, which has measure zero, the deterministic components contribute nothing to the evolutions of the processes, so we need only consider the stochastic components, *i.e.*, on the set in question  $d \ln X_t = dX_t/X_t$  and  $d \ln c_t = dc_t/c_t$ .

which, solving for  $\omega$  using  $z^* = \left(\frac{a}{1-a}\right) \left(\frac{\ln \kappa}{\kappa-1}\right)^{\phi-1} p^*$ , implies

$$\omega = \frac{1}{1 + \frac{(1-a)K(\kappa)}{a\phi(\kappa-1)p^*}} \in (0, 1) \quad (24)$$

where we have dropped the subscript- $t$  on  $\omega$  because it is independent of the time at which development occurs.

Equation (24) specifies the ‘‘resource division parameter,’’ *i.e.*, the fraction, at the development boundary, of the *natural growth* in the non-housing capital stock / conversion rate ratio that actually contributes to growth in this ratio, with the complement applied to increasing the stock of housing capital.

The non-housing capital stock process therefore evolves as

$$Y_{0,t} = X_{0,t} M_{0,t}^{\omega-1} \quad (25)$$

where the first term quantifies the effect of natural growth on the stock and the second term accounts for the diversion of a fraction  $1 - \omega$  of the natural growth in non-housing resources to the housing sector. Equation (18) and the definition of  $\omega$  also imply that  $d \ln Q_t = \frac{\omega}{\phi} d \ln x_t$  at the development boundary, so we can write the housing capital stock process as

$$Q_{0,t} = e^{-\delta t} M_{0,t}^{\omega/\phi} \quad (26)$$

where the first term quantifies the effect of natural depreciation on the stock and the second term accounts for new development.

The two together imply that aggregate consumption  $C_t = Y_t^a Q_t^{1-a}$  evolves as

$$C_{0,t} = e^{-(1-a)\delta t} X_{0,t}^a M_{0,t}^b \quad (27)$$

where  $b = \left(a + \frac{1-a}{\phi}\right) \omega - a$ . Note that the previous equation differs from that which would hold absent the conversion technology only by the term  $M_{0,t}^b$ , so this term captures the impact of the existence of the technology, *i.e.*, of new housing development.



### 3.3.2 The Price of the Second Good and the Pricing Kernel

It is trivial, now that we have  $Y_t$  and  $Q_t$  in terms of  $\zeta_t$  and its history, to calculate the price of housing services and the pricing kernel in terms of the same.

The scaled price  $p_t = P_t/c_t = \left(\frac{1-a}{a}\right) y_t/Q_t$ , so using  $x_{0,t} = e^{-\phi\delta t} \zeta_{0,t}$

$$p_{0,t} = e^{-(\phi-1)\delta t} \zeta_{0,t} M_{0,t}^{\Psi-1} \quad (28)$$

where  $\Psi = (\phi - 1)\omega/\phi$ . Unscaled, the price evolves as  $P_{0,t} = e^{\delta t} X_{0,t} M_{0,t}^{\Psi-1}$ , where the  $X_{0,t}$  represents the effect of changes in housing demand induced by productivity shocks to production of the numeraire good, and  $e^{\delta t}$  and  $M_{0,t}^{\Psi-1}$  represent the effect of changes in supply due to depreciation and new construction, respectively.

The pricing kernel is given by  $\xi_t^Y = Y_{0,t}^{-\gamma_Y} Q_{0,t}^{1-\gamma_Q}$ , so using our formulae for the capital stock processes we can rewrite the pricing kernel in terms of the underlying processes as

$$\xi_{0,t}^Y = \underbrace{X_{0,t}^{-\gamma_Y}}_{\substack{\text{impact of} \\ \text{natural growth of} \\ \text{non-housing capital}}} \underbrace{M_{0,t}^{\gamma_Y(1-\omega)}}_{\substack{\text{impact of} \\ \text{resources diverted to} \\ \text{housing sector}}} \underbrace{e^{-(1-\gamma_Q)\delta t}}_{\substack{\text{impact of} \\ \text{natural depreciation} \\ \text{of housing stock}}} \underbrace{M_{0,t}^{(1-\gamma_Q)\omega/\phi}}_{\substack{\text{impact of} \\ \text{new housing} \\ \text{development}}}. \quad (29)$$

The first two terms quantify the impact of changes in the level of non-housing consumption on the marginal utility of non-housing consumption, from natural growth in the non-housing sector and the diversion of resources to the housing sector, respectively. The third and fourth terms quantify the impact of changes in the level of housing consumption, from depreciation and new development. Simplifying yields

$$\xi_{0,t}^Y = e^{-(1-\gamma_Q)\delta t} X_{0,t}^{-\gamma_Y} M_{0,t}^{\Omega} \quad (30)$$

where  $\Omega = (1 - \omega)\gamma_Y + (1 - \gamma_Q)\omega/\phi$ .<sup>10</sup> Equation (30) has important asset pricing implications,

<sup>10</sup> The kernel has an alternative, “observable” representation in terms of aggregate consumption of the numeraire good and the price of the second good. Using the definition of  $\gamma_Y$  and  $\gamma_Q$ , we have that  $\xi_{0,t}^Y = Y_{0,t}^{-\gamma_Y} Q_{0,t}^{1-\gamma_Q} = Y_{0,t}^{-\gamma_C} (Q_{0,t}/Y_{0,t})^{(1-a)(1-\gamma_C)}$ . Then using the fact that the price of housing services is proportional to the ratio of non-housing and housing consumption,  $P_{0,t} = Y_{0,t}/Q_{0,t}$ , we can express the kernel as

$$\xi_{0,t}^Y = Y_{0,t}^{-\gamma_C} P_{0,t}^{-(1-a)(1-\gamma_C)}.$$

Here the first term is the kernel in the standard Lucas economy, and reflects agents’ concern for (numeraire) consumption

which depend qualitatively on the sign of  $\Omega$ , which determines whether the net impact of diverting resources from the non-housing sector to the housing sector increases ( $\Omega > 0$ ) or decreases ( $\Omega < 0$ ) the representative consumer's marginal utility from numeraire consumption. Pro-cyclicality of the price of non-housing capital (a high price/earnings ratio for the non-housing sector when developers are increasing the supply of housing) requires that  $\Omega > 0$ , which places an upper bound on  $\gamma_C$ , the consumer's risk aversion over aggregate consumption.

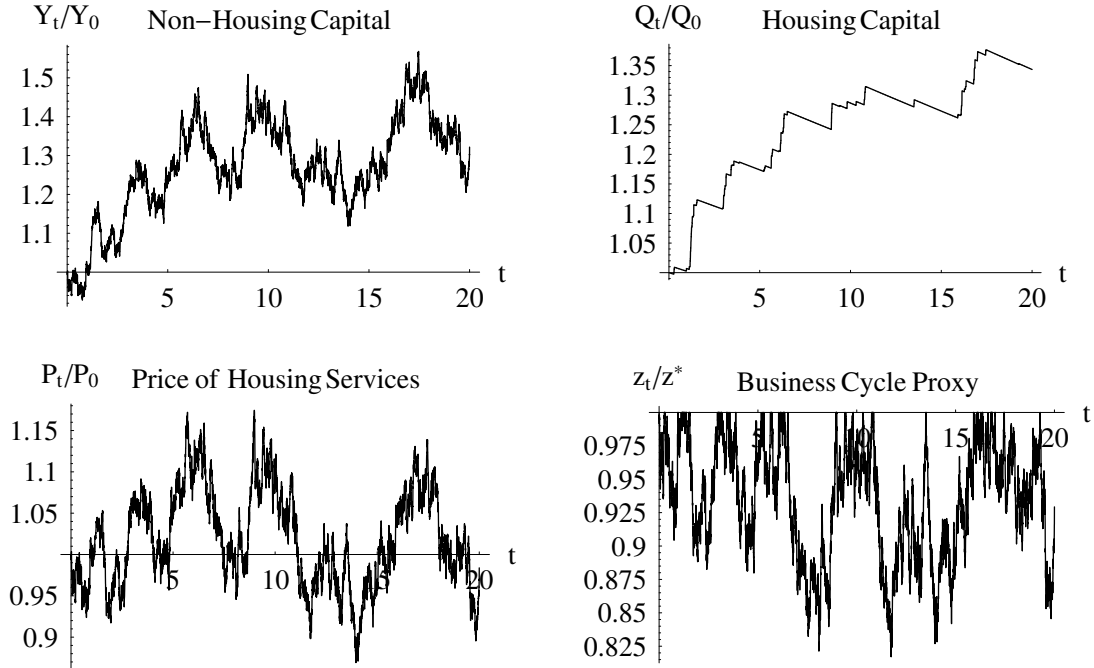
Note that the evolution of the log-processes given in equations (28) and (30) depend on  $z_t$  (because  $M_t = \zeta_t z^*/z_t$ ) but not on their own levels, which implies the multiplicative investment strategy of section 3.2 used to generate these processes. This motivates our formal equilibrium analysis, and in particular the explicit investment strategy we will hypothesize in section 4. Before formalizing the equilibrium analysis, however, it is worthwhile to consider the evolution of the aggregate processes, and in particular the capital stock (consumption) processes,  $Y_t$  and  $Q_t$ , the price of housing services,  $P_t$ , and the business cycle proxy,  $z_t$ .

### 3.4 Sample Paths

Figure 1 shows sample paths of the two capital stocks, the price of housing services, and the business cycle proxy  $z_t/z^*$ . The non-housing capital stock (upper left) evolves as a geometric Brownian process partially reflected at a stochastic barrier, the level of which depends both on the history of the process and on the evolution of the inverse productivity process,  $c_t$ . The housing capital stock, which grows in response to development and decays in its absence, depreciates exponentially off the set on which it is increasing, and on the set on which it is increasing the increments look like those of the maximum of a geometric Brownian process. The price process (lower left), which is proportional to the ratio of the two stock processes, appears stationary over long intervals, even though it is not, as the growth of the two stock processes are highly correlated (endogenously). The business cycle proxy  $z_t/z^*$  (lower right) evolves as a reflected geometric Brownian process. When this ratio is close to one the housing stock is likely to expand in the near future. We associate these times with economic expansion. When the ratio is significantly less than one it may be years before significant development of new housing capacity. We identify these times with recessions.

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risk, while the second term reflects their concern for "composition risk," fluctuations in the real consumption of housing services relative to other goods.



**Figure 1: Capital Stocks, Price of Housing Services, and the Business Cycle Proxy**

Sample paths for the capital stock processes,  $Y_t$  and  $Q_t$  (upper left and upper right, respectively), the price of housing services,  $P_t$  (lower left), and the business cycle proxy  $z_t/z^*$  (lower right). Parameters used to generate the paths are  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$  for the technological shocks, and  $\phi = 1.5$ ,  $\delta = 0.01$  and  $\omega = 0.893$  for the conversion technology (consistent, we will see later, with  $K(q) = q^\phi$ ,  $\gamma_C = 1.5$  and  $a = 2/3$ ).

## 4 Equilibrium

At this point we will formalize the argument given heuristically in the previous section, explicitly calculating the optimal development strategy. The equilibrium concept we employ is competitive equilibrium, and we assume agents have rational expectations.

Demonstrating an equilibrium amounts to solving a fixed-point problem. We will begin by hypothesizing an explicit investment strategy that if “imposed” on all agents determines the evolution of the capital stock processes, as functions of the evolution of the exogenous processes, the natural growth of non-housing capital and the productivity of housing capital of a given vintage. This strategy will consequently determine the evolution of the price of housing services and the pricing kernel, which depend on the consumption processes and the preferences of the representative consumer. Value maximization then defines a mapping from the evolution of the price of housing

services and the pricing kernel to investment strategies. An equilibrium strategy is a fixed point in the composition mapping, which takes exogenously imposed strategies to their optimal responses.

The statement of the strategy hypothesis is facilitated by introducing the following notation. Let  $\mathbf{Z}_t \equiv (\ln X_t, -\ln c_t)'$  denote the vector of the log-natural growth of non-housing capital and the log-productivity of vintage- $t$  housing capital processes, and let  $\mathbf{M}$  and  $\Sigma$  denote the associated drift and variance-covariance matrices,

$$\mathbf{M} \equiv \frac{\mathbf{E}[d\mathbf{Z}_t]}{dt} = \begin{pmatrix} \mu_X - \frac{\sigma_X^2}{2} \\ -\mu_c + \frac{\sigma_c^2}{2} \end{pmatrix} \quad (31)$$

$$\Sigma \equiv \frac{\mathbf{E}[d\mathbf{Z}_t d\mathbf{Z}_t']}{dt} = \begin{pmatrix} \sigma_X^2 & -\rho\sigma_X\sigma_c \\ -\rho\sigma_X\sigma_c & \sigma_c^2 \end{pmatrix}. \quad (32)$$

We will also let  $\mathbf{e}_X = (1, 0)'$ ,  $\mathbf{1} = (1, 1)'$ , and  $\boldsymbol{\beta} \equiv \frac{\Sigma\mathbf{1}}{\mathbf{1}'\Sigma\mathbf{1}}$ , and use

$$\mu_\zeta \equiv \frac{\mathbf{E}[d\zeta_t/\zeta_t]}{dt} = \mathbf{1}'\mathbf{M} + \frac{\mathbf{1}'\Sigma\mathbf{1}}{2} + \phi\delta \quad (33)$$

$$\sigma_\zeta \equiv \left( \frac{\mathbf{E}[(d\zeta_t/\zeta_t)^2]}{dt} \right)^{1/2} = \sqrt{\mathbf{1}'\Sigma\mathbf{1}} \quad (34)$$

to denote the drift and volatility of  $\zeta_t \equiv z_0 e^{\phi\delta t} X_{0,t}/c_{0,t}$ .

## 4.1 Strategy Hypothesis

**The Strategy Hypothesis.** *Suppose that both*

1. *housing sites' log-capacities are distributed uniformly between  $\ln q_t^{\min}$  and  $\ln \kappa q_t^{\min} = \ln q_t^{\max}$ , where  $\kappa$  is the (unique) solution in  $x > 1$  to*

$$\frac{(\eta-1)K'(x)}{K(x)} = \frac{\eta}{x-1} - \frac{\eta\phi - \eta - \phi}{x^{(\eta-1)(\phi-1)} - x} \quad (35)$$

where

$$\Psi\eta = \alpha + \mathbf{e}'_X \mathbf{b} + \left( 1 - (1 - \gamma_C) \left( a + \frac{1-a}{\phi} \right) \right) \omega \quad (36)$$

with  $\Psi = (\phi - 1)\omega/\phi$ ,  $\mathbf{b} = a(1 - \gamma_C)(\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta})\mathbf{1})$ , and  $\alpha > 0$  and  $\omega \in (0, 1)$  solve,

respectively, the following quadratic equations:

$$(\mathbf{1}'\mathbf{M} + \phi\delta)\alpha + \left(\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{2}\right)\alpha^2 = \lambda + (1 - \gamma_C) \left((1 - a) + a(\mathbf{e}'_X\boldsymbol{\beta})\phi\right)\delta - \mathbf{b}'\mathbf{M} - \frac{\mathbf{b}'\boldsymbol{\Sigma}\mathbf{b}}{2} \quad (37)$$

$$(1 - a)(\Psi\eta - 1)\omega = \phi a\Psi\eta(1 - \omega) \quad (38)$$

and,

2. the price of housing services, relative to the conversion rate, is “not too high,” satisfying

$$p_t \leq q_t^{\phi-1} p^* \text{ where}$$

$$p^* = \frac{K(\kappa)}{\left(1 - \frac{1}{\Psi\eta}\right)(\kappa - 1)}. \quad (39)$$

Then firms optimally follow a capacity-dependent trigger strategy in the price/conversion-rate ratio, with a firm with capacity  $q_t$  optimally redeveloping to capacity  $\kappa q_t$  at  $\tau = \min\{s > t | P_s/c_s = q_s^{\phi-1} p^*\}$ .

## 4.2 Project Value

Consider the value of a particular project, supposing all other developers are “constrained” to follow the hypothesized strategy, denoting the current time by 0 without loss of generality. Letting  $T = \min_{t>0}\{z_t = z^*\}$  denote the stopping time for some developer first adding capacity under the hypothesized strategy, we can write the value of an unconstrained developer as

$$V(q_0, P_0, c_0, z_0) = \mathbf{E}_0 \left[ \int_0^T e^{-\lambda t} \xi_{0,t}^Y q_t P_t dt + e^{-\lambda T} \xi_{0,T}^Y V(q_T, P_T, c_T, z^*) \right]. \quad (40)$$

The first term on the right hand side of the previous equation, using the fact that  $\zeta_t = z_t < z^*$  and consequently  $M_t = z^*$  for all  $t < T$ , may be written as

$$\begin{aligned} \mathbf{E}_0 \left[ \int_0^T e^{-\lambda t} \xi_{0,t}^Y q_t P_t dt \right] &= \mathbf{E}_0 \left[ \int_0^T e^{-\lambda t} \left( e^{-(1-\gamma_Q)\delta t} X_{0,t}^{-\gamma_Y} \right) \left( e^{-\delta t} q_0 \right) \left( e^{\delta t} X_{0,t} P_0 \right) dt \right] \\ &= q_0 P_0 \left( 1 - \mathbf{E}_0 \left[ e^{-\hat{\lambda} T} X_{0,T}^{1-\gamma_Y} \right] \right) \pi \end{aligned} \quad (41)$$

where  $\hat{\lambda} \equiv \lambda + (1 - \gamma_Q)\delta$ , and  $\pi \equiv \mathbf{E}_0 \left[ \int_0^\infty e^{-\hat{\lambda} t} X_{0,t}^{1-\gamma_Y} dt \right]$  is the perpetuity factor for revenues derived from buildings in the absence of new supply effects (*i.e.*, in a world in which it is impossible

to develop additional housing capacity).

The second term, using the degree- $\phi$  homogeneity of  $V$  in  $q$  and  $P^{1/(\phi-1)}$  jointly, and that  $P_{0,T}/c_{0,T} = e^{-(\phi-1)\delta T} \zeta_{0,T}$  and  $\zeta_{0,T} = z^*/z_0$ , is

$$\begin{aligned} \mathbf{E}_0 \left[ e^{-\lambda T} \xi_{0,T}^Y c_{0,T} V \left( e^{-\delta T} q_0, e^{-(\phi-1)\delta T} \zeta_{0,T} P_0, c_0, z^* \right) \right] \\ = \mathbf{E}_0 \left[ e^{-(\lambda+\phi\delta)T} \xi_{0,T}^Y c_{0,T} \right] V \left( q_0, \left( \frac{z^*}{z_0} \right) P_0, c_0, z^* \right). \end{aligned} \quad (42)$$

Equations (40) through (42) taken together, after explicitly evaluating  $\mathbf{E}_0 \left[ e^{-\lambda T} X_{0,T}^{1-\gamma_Y} \right]$ ,  $\pi$  and  $\mathbf{E}_0 \left[ e^{-(\lambda+\phi\delta)T} \xi_{0,T}^Y c_{0,T} \right]$ , yield the following proposition. The proofs of all propositions, in an effort to avoid excessive expositional digression, are left for the appendix.

**Proposition 4.1.** *The value of a project with current capacity  $q$  to an unconstrained developer is*

$$V(q_0, P_0, c_0, z_0) = q_0 P_0 \pi + \left( \frac{z_0}{z^*} \right)^{\Psi\eta + \Omega} \left[ V \left( q_0, \left( \frac{z^*}{z_0} \right) P_0, c_0, z^* \right) - q_0 \left( \frac{z^*}{z_0} \right) P_0 \pi \right] \quad (43)$$

where  $\Psi\eta$  is given in the Strategy Hypothesis (equation (36)),  $\Omega = (1-\omega)\gamma_Y + (1-\gamma_Q)\omega/\phi$  and

$$\pi = \frac{1}{\lambda - \mu} \quad (44)$$

where  $\mu = (1-\gamma_Y)(\mu_X - \gamma_Y \sigma_X^2/2) - (1-\gamma_Q)\delta$ .

Note that the perpetuity factor on the previous equation would be the price/earnings ratio of non-housing capital if there were no conversion technology, *i.e.*, in a Lucas economy in which the representative agent has a coefficient of relative risk aversion of  $\gamma_Y$  and the rate of time preference is adjusted to account for the impact of depreciation in the housing sector on the marginal utility of non-housing consumption.

### 4.3 Value at the Development Boundary

Note that in equation (43) the value of building to an unconstrained developer only depends on the developer's strategy through  $V \left( q, \left( \frac{z^*}{z} \right) p, c, z^* \right)$ , the value at the time other developers start to build new capacity. Therefore, in order to maximize the site's value, we only need to find the strategy that maximizes its value at the constrained firms' development boundary.

We may similarly decompose the scaled value of the unconstrained developer's project at the constrained development boundary  $z^*$ . Letting  $\zeta^*$  denote the level of  $\zeta$  at which the unconstrained

developer optimally develops and  $\tau \equiv \min_{s>0}\{\zeta_{t+s} = \zeta^*\}$ , we can write the scaled value of the unconstrained firm at the development boundary  $z^*$  as

$$V(q_t, P_t, c_t, z^*) = \mathbf{E}_t \left[ \int_0^\tau e^{-\lambda s} \xi_{t,s}^Y q_{t+s} P_{t+s} ds + e^{-\lambda \tau} \xi_{t,\tau}^Y V(q_{t+\tau}, P_{t+\tau}, c_{t+\tau}, z^*) \right] \quad (45)$$

The first term on the right hand side of the previous equation is

$$\begin{aligned} \mathbf{E}_t \left[ \int_0^\tau e^{-\lambda s} \xi_{t,s}^Y q_{t+s} P_{t+s} ds \right] &= q_t P_t \mathbf{E}_t \left[ \int_0^\tau e^{-\lambda s} \left( e^{-(1-\gamma_Q)\delta t} X_{t,s}^{-\gamma_Y} M_{t,s}^\Omega \right) X_{t,s} M_{t,s}^{\Psi-1} ds \right] \\ &= q_t P_t \mathbf{E}_t \left[ \left( \int_0^\infty - \int_\tau^\infty \right) e^{-\lambda t} X_{t,s}^{1-\gamma_Y} M_{t,s}^{\Omega+\Psi-1} ds \right] \\ &= q_t P_t \left( 1 - \zeta_{t,\tau}^{\Omega+\Psi-1} \mathbf{E}_t \left[ e^{-\lambda \tau} X_{t,\tau}^{1-\gamma_Y} \right] \right) \Pi^{II} \end{aligned} \quad (46)$$

where  $\Pi^{II} \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-(\lambda+\delta)s} \xi_{t,s}^Y P_{t,s} ds | z_t = z^* \right]$  is the perpetuity factor for revenues derived from housing capital at a time when the stock of housing capital is increasing.

The second term on the right hand side of equation (45), using the homogeneity of  $V$  in  $q$  and  $P^{1/(\phi-1)}$ , and  $P_{t,\tau}/c_{t,\tau} = e^{-(\phi-1)\delta\tau} \zeta_{t,\tau}^\Psi$ , is

$$\mathbf{E}_t \left[ e^{-\lambda \tau} \xi_{t,\tau}^Y c_{t,\tau} V \left( e^{-\delta\tau} q_t, e^{-(\phi-1)\delta\tau} \zeta_{t,\tau}^\Psi P_t, c_t, z^* \right) \right] = \mathbf{E}_t \left[ e^{-(\lambda+\phi\delta)\tau} \xi_{t,\tau}^Y c_{t,\tau} \right] V \left( q_t, \zeta_{t,\tau}^\Psi P_t, c_t, z^* \right). \quad (47)$$

Equations (45) through (47), after evaluating  $\mathbf{E}_t \left[ e^{-\lambda \tau} X_{t,\tau}^{1-\gamma_Y} \right]$ ,  $\Pi^{II}$  and  $\mathbf{E}_t \left[ e^{-(\lambda+\phi\delta)\tau} \xi_{t,\tau}^Y c_{t,\tau} \right]$  explicitly, yield the following proposition.

**Proposition 4.2.** *At the investment barrier  $z^*$  the value of a project to an unconstrained developer is given by*

$$V(q_t, P_t, c_t, z^*) = q_t P_t \Pi^{II} + \left( \frac{\zeta_t}{\zeta^*} \right)^{\Psi\eta} \left[ V \left( q_t, \left( \frac{\zeta_t}{\zeta^*} \right)^\Psi P_t, c_t, z^* \right) - q_t \left( \frac{\zeta_t}{\zeta^*} \right)^\Psi P_t \Pi^{II} \right] \quad (48)$$

where  $\eta$  and  $\Psi$  are given in the Strategy Hypothesis,

$$\Pi^{II} = \left( \frac{\Psi\eta + \Omega - 1}{\Psi\eta - \Psi} \right) \pi, \quad (49)$$

and  $\pi$  and  $\Omega$  are given in Proposition 4.1.

The first term in equation (48) represents the value of assets-in-place, the expected value of the appropriately discounted future revenues from the project's existing capital stock. The second term accounts for real option value, the value of the ability to increase production in the future, which

represents a technological rent resulting from 1) the adjustment costs, which limit competition on the intensive margin, and 2) the scarcity of sites, which limits competition on the extensive margin.

#### 4.4 Optimal Development Strategy

Optimality and feasibility of the investment strategy requires that the value matching and smooth pasting conditions hold, *i.e.*, that project value is continuous and differentiable in the underlying control across development. Assuming a firm initially endowed with a site developed to unit capacity at time- $t$  optimally chooses to redevelop the site to a multiple of  $q$  times its existing intensity at time  $t + \tau$ , then

$$V(e^{-\delta\tau}, P_{t+\tau}, c_{t+\tau}, z^*) = V(e^{-\delta\tau}q, P_{t+\tau}, c_{t+\tau}, z^*) - c_{t+\tau}\Pi^I K(e^{-\delta\tau}q) \quad (50)$$

$$V_\zeta(e^{-\delta\tau}, P_{t+\tau}, c_{t+\tau}, z^*) = V_\zeta(e^{-\delta\tau}q, P_{t+\tau}, c_{t+\tau}, z^*). \quad (51)$$

Using the inherited joint homogeneity of  $v$  in  $q$  and  $p^{1/(\phi-1)}$  and the homogeneity of  $K$ , together with the fact that  $q_{t+\tau} = e^{-\delta\tau}$  and  $p_{t,\tau} = e^{-(\phi-1)\delta\tau}\zeta_{t,\tau}^\Psi$ , the previous equations imply

$$v\left(1, \zeta_{t,\tau}^\Psi p_t\right) = v\left(q, \zeta_{t,\tau}^\Psi p_t\right) - \Pi^I K(q) \quad (52)$$

$$v_\zeta\left(1, \zeta_{t,\tau}^\Psi p_t\right) = v_\zeta\left(q, \zeta_{t,\tau}^\Psi p_t\right) \quad (53)$$

or, using the functional form for  $v$  implied by (48) and  $\frac{dp}{d\zeta}|_{z=z^*} = \Psi p/\zeta$ , that

$$\zeta_{t,\tau}^\Psi p \Pi^{II} + a_1 \zeta_{t,\tau}^{\Psi\eta} = q \left(\frac{\zeta^*}{\zeta}\right)^\Psi p \Pi^{II} + a_q \left(\frac{\zeta^*}{\zeta}\right)^{\Psi\eta} - \Pi^I K(q) \quad (54)$$

$$\zeta_{t,\tau}^\Psi p \Pi^{II} + \eta a_1 \zeta_{t,\tau}^{\Psi\eta} = q \zeta_{t,\tau}^\Psi p \Pi^{II} + \eta a_q \zeta_{t,\tau}^{\Psi\eta} \quad (55)$$

where  $a_x = \left(v\left(x, (\zeta^*/\zeta)^\Psi p\right) - x (\zeta^*/\zeta)^\Psi p \Pi^{II}\right)$ . Solving these immediately yields

$$a_1 - a_q = \frac{\Pi^I K(q)}{(\eta - 1)} \left(\frac{\zeta}{\zeta^*}\right)^{\Psi\eta} \quad (56)$$

$$\zeta_{t,\tau}^\Psi = \frac{\eta \Pi^I K(q)}{(\eta - 1) \Pi^{II} (q - 1) p}. \quad (57)$$

Finally, the homogeneity of  $v$ , which implies  $v(q, p) = q^\phi v(1, p/q^{\phi-1})$ , in conjunction with the functional form for  $v$  given in equation (48), and the fact that  $(\zeta/\zeta^*)^\Psi$  is proportional to  $p$ ,



means that  $a_q = q^{\eta+\phi-\eta\phi} a_1$ , or

$$a_1 = \frac{a_1 - a_q}{1 - q^{\eta+\phi-\eta\phi}}, \quad (58)$$

which together with equations (56) and (57) yields

$$a_1 = \left( \frac{\Pi^I K(q)}{(\eta-1)(1-q^{\eta+\phi-\eta\phi})} \right) \left( \frac{(\eta-1)\Pi^{II}(q-1)p}{\eta\Pi^I K(q)} \right)^\eta. \quad (59)$$

The developer maximizes project value, so the optimal strategy is the one that maximizes  $a_1$  over the choice variable  $q$ , *i.e.*, to redevelop to  $\kappa$  times existing capacity where

$$\kappa = \arg \max_{q>1} \left\{ \frac{(q-1)^\eta}{(1-q^{\eta+\phi-\eta\phi}) K(q)^{\eta-1}} \right\}. \quad (60)$$

**Proposition 4.3.** *Equation (60) defines  $\kappa < \infty$  uniquely, and agrees with that given in equation (35) in the Strategy Hypothesis.*

We can also characterize the optimal development strategy in terms of the price of housing services relative to the conversion rate. Solving for this price-cost ratio at the time of development,  $p_{t+\tau} = e^{-(\phi-1)\delta\tau} \zeta_{t,\tau}^\Psi p_t$ , using equation (57) and  $q_{t+\tau} = e^{-\delta\tau} q_t$ , yields

$$p_{t+\tau} = q_{t+\tau}^{\phi-1} \left( \frac{\eta\Pi^I K(\kappa)}{(\eta-1)\Pi^{II}(\kappa-1)} \right), \quad (61)$$

which, after evaluating  $\Pi^I$  and  $\Pi^{II}$  explicitly, yields the following proposition.

**Proposition 4.4.** *The optimal investment strategy is the capacity-dependent trigger strategy in  $p$ , the price of the numeraire consumption good relative to the conversion rate, given in the Strategy Hypothesis: develop to capacity  $\kappa q_{\tau^*}$  at  $\tau^* \equiv \min_{s>t} \{p_s = q_s^{\phi-1} p^*\}$  where*

$$p^* = \frac{K(\kappa)}{\left(1 - \frac{1}{\Psi\eta}\right)(\kappa-1)}. \quad (62)$$

#### 4.5 Q-Theoretic Interpretation of the Equilibrium

Note that under the equilibrium strategy the value of revenues derived from a new unit of housing capital is  $P_t \Pi^{II}$ , the price of housing services times the unit value of revenues derived from housing

services. The cost of a new unit of housing capacity is  $NCR_t \Pi^I$ , the net unit conversion rate times the unit cost of non-housing capital. The price of housing services/net conversion rate ratio, after substituting for  $NCR_t$  and  $P_t$  and using the fact that at the time investment takes place  $Y_t/c_t Q_t^\phi = z^*$ , is given by

$$\frac{P_t}{NCR_t} = \frac{\Psi\eta}{\Psi\eta - 1}. \quad (63)$$

Then using the fact that  $\Pi^{II} = \left(\frac{\Psi\eta + \Omega - 1}{\Psi\eta - \Psi}\right) \pi$ , as given in equation (49), and

$$\Pi^I = \left(\frac{\Psi\eta + \Omega - 1}{\Psi\eta - 1}\right) \pi, \quad (64)$$

calculated in the proof of propositions 4.4, we have that the value/cost ratio of net new housing capital, *i.e.*, Tobin's  $Q$  of new housing capital, is equal to

$$\frac{P_t \Pi^{II}}{NCR_t \Pi^I} = \frac{\eta}{\eta - 1}. \quad (65)$$

That is, the irreversible nature of the building decision leads developers to delay investing until the value of additional new capacity exceeds the cost of development by a factor of  $1 + 1/(\eta - 1)$ .

## 4.6 Explicit Valuation

Combining the results of Propositions 4.1, 4.2 and 4.4 produces a simple, explicit formula for a building's price/earnings ratio (inverse capitalization rate), which depends only on 1) the building's relative size (developmental density), and 2) how close the price of housing services/conversion rate ratio is to the level that would induce the developer with the smallest project to alter capacity immediately, *i.e.*, to the business cycle proxy  $z_t/z^*$ . Note also that price/earnings ratios, and consequently the following proposition, are independent of numeraire.

**Proposition 4.5.**  $\Pi^{II}\left(\frac{q}{q}, z/z^*\right) \equiv V\left(q, P, c, \underline{q}\right)/qP$ , the price/earnings ratio of a building with capacity  $q$  when the least developed site has capacity  $\underline{q}$ , is given by

$$\Pi^{II}\left(\frac{q}{q}, \frac{z}{z^*}\right) = \pi + \left(\frac{z}{z^*}\right)^{\Psi\eta + \Omega - 1} \left( \left(\Pi^{II} - \pi\right) + \left(\frac{q}{q}\right)^{(\eta-1)(\phi-1)} \left(\frac{\kappa-1}{\eta(1-\kappa^{\eta+\phi-\eta\phi})}\right) \Pi^{II} \right) \quad (66)$$

where  $\kappa$  and  $\eta$  are given in the strategy hypothesis,  $\pi$  and  $\Omega$  are given in proposition 4.1, and  $\Pi^{II}$

is given in proposition 4.2.

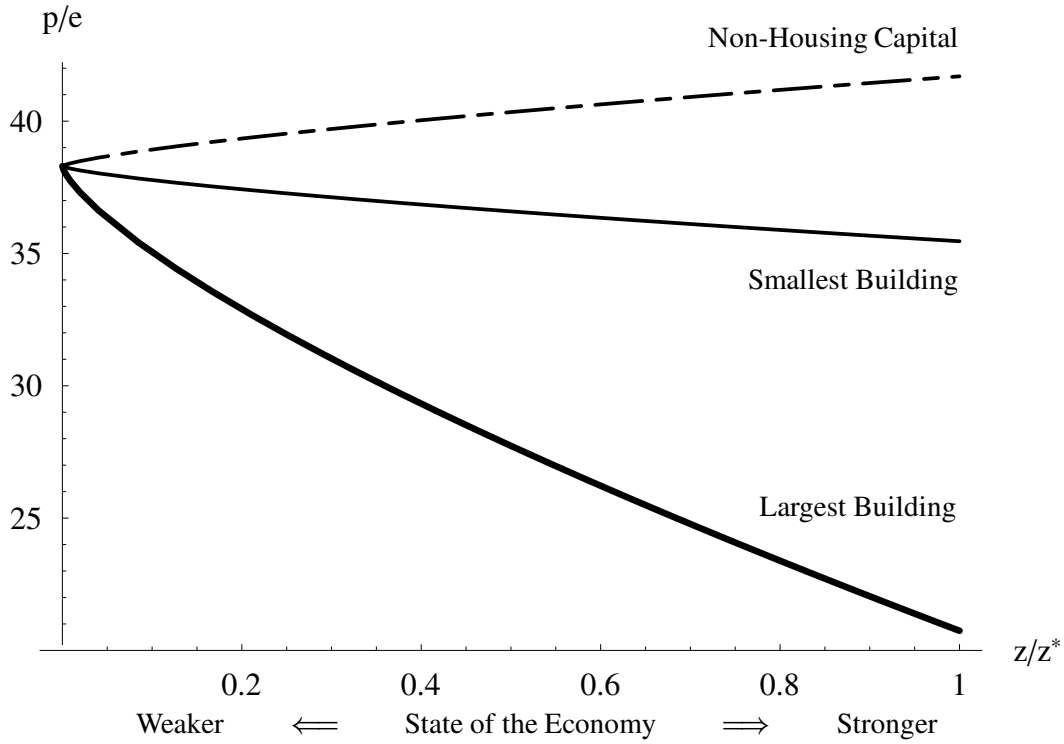
In the previous equation, the first term is the unit value of revenue in an economy with no conversion technology. The second term corrects for other firms' supply responses, which come sooner when economic conditions are strong, and have a negative impact on output prices, and consequently on the unit value of revenue. These two terms together represent the unit value of revenues from assets-in-place. The third term quantifies the value of the firm's growth options, *i.e.*, the value of its ability to increase capacity in the future, and represents economic rents that accrue to the scarcity of the site.

Figure 2 shows inverse capitalization-rates (*i.e.*, price/earnings ratios) for the largest and smallest buildings in the economy, as a function of the business cycle proxy  $z_t/z^*$ . It also shows the inverse cap-rate for non-housing capital (*i.e.*, the dividend yield on apple trees), given by

$$\Pi^I \left( \frac{z}{z^*} \right) = \pi + \left( \frac{z}{z^*} \right)^{\Psi\eta + \Omega - 1} \left( \Pi^I - \pi \right). \quad (67)$$

The difference in the price/earnings ratios of non-housing capital and buildings is always positive, and pro-cyclical. Housing assets-in-place are less exposed to the primary risk in the economy, the natural growth of non-housing capital, and are less exposed in expansions than in recessions. While the price of housing services is highly correlated with non-housing production, real estate development is pro-cyclical and the price impact of new supply acts as a countervailing force, reducing the exposure of housing assets-in-place to risk in the natural growth of non-housing capital. The difference in the p/e ratios of small and large buildings is also always positive, *i.e.*, large buildings have higher capitalization rates than small buildings, because a smaller building's value has a larger real options component, which does not generate current revenues. Because the real option component is pro-cyclical, the difference in the p/e ratios of small and large buildings is also always pro-cyclical, *i.e.*, the cap-rate spread between recent construction and older buildings is increasing in the strength of the economy. This cap-rate spread should, consequently, predict new development.

We can also use equation (66) to consider the contribution of growth options to a building's value in greater detail. The contribution of growth options to total value can be calculated by dividing the third term of equation (66) by the total. The results of doing so are shown in figure 3 as a function of the size of the building (log relative size) for two different states of the economy, a "strong" economy, with  $z_t$  at the 95% level of its stationary distribution (top curve), and a "weak" economy,



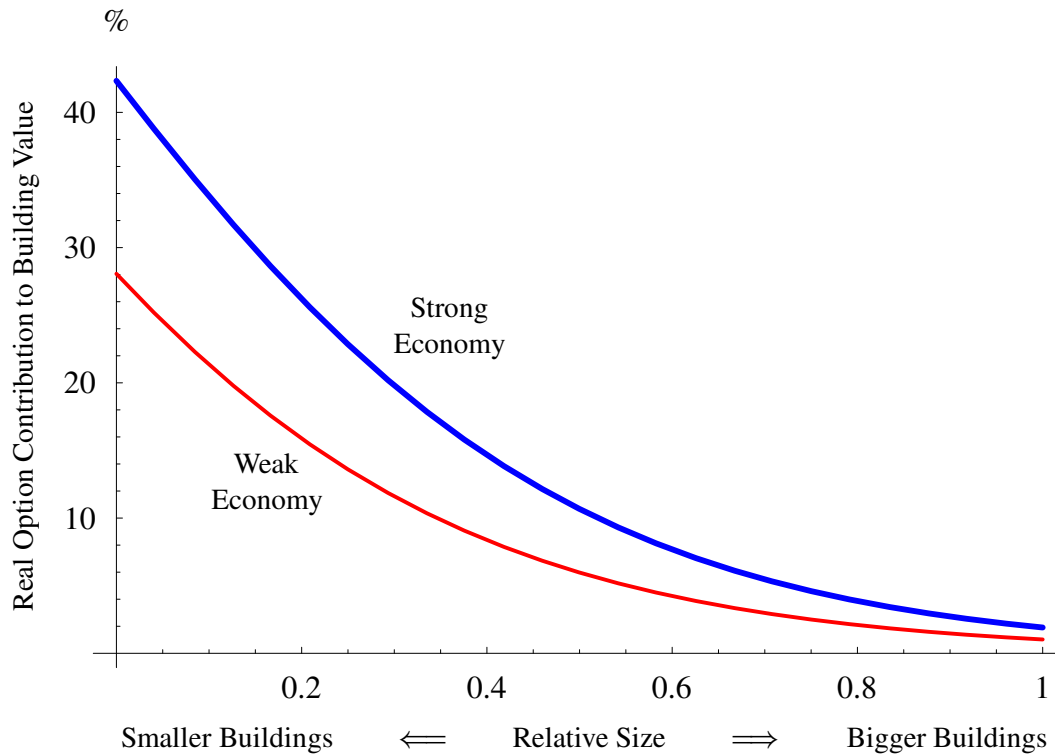
**Figure 2: Price/Earnings Ratios**

Price/earnings ratios for non-housing capital (dashed line, top) and the smallest and largest building (respectively solid line, middle, and bold solid line, bottom), as a function of the business cycle proxy  $z_t$ . Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_c = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$  (giving  $\omega = 0.882$ ).

with  $z_t$  at the 5% level (bottom curve).<sup>11</sup> Not surprisingly, growth options are a more important component of smaller buildings' values, as these sites will be redeveloped sooner, so the value of the revenues generated from future capacity is discounted less. Growth options' contribution to value is also pro-cyclical, again because strong economic conditions make near term development more likely, and reduce the required discount on the associated revenues. While real options are relatively unimportant for the value of large buildings, a change in zoning that prevents the owner of a small building from redeveloping the site can reduce the building's value by more than two fifths. Overall,

<sup>11</sup> Brownian motion, reflected from above, has an exponential distribution with exponent  $2\mu/\sigma^2$ . The stationary distribution of  $z/z^*$ , which is a reflected geometric Brownian process, is therefore given by  $dv_{z/z^*}(x) = P(\ln z/z^* = \ln x)d \ln x = (2(\mu_\xi - \sigma_\xi^2/2)/\sigma_\xi^2) \exp\left(\left(2(\mu_\xi - \sigma_\xi^2/2)/\sigma_\xi^2 - 1\right) \ln x\right) dx/x = (2\mu_\xi/\sigma_\xi^2 - 1) x^{2\mu_\xi/\sigma_\xi^2 - 2} dx$ . The  $m$ -level of the stationary distribution, defined as the  $z_m$  such that  $\int_0^{z_m/z^*} dv_{z/z^*}(z) = m$ , is given by  $z_m = m^{1/(2\mu_\xi/\sigma_\xi^2 - 1)} z^*$ .

real option premia represent a significant component of the aggregate value of developed real estate, contributing 8% of the aggregate value in the lower curve, and more than 11% of aggregate value in the upper curve.



**Figure 3: Contribution of Growth Options to Total Value**

Real option value as a percentage of total project value, as a function of relative size (smallest building left, largest right). Top curve (bold, blue) shows an expansionary economy, with  $z_t$  at the 95% level of the stationary distribution, while the bottom curve (thin, red) shows a weak economy, with  $z_t$  at the 5% level. Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

## 5 Properties of the Equilibrium

Before delving into implications for economic growth and asset pricing, we will first consider some basic properties of the equilibrium derived in the previous section. These basic properties include the expected time between successive capacity adjustments, and the degree of heterogeneity in

building sizes. We are interested in how the magnitude of adjustment costs impacts the frequency of capacity adjustments and economic heterogeneity generally, *i.e.*, beyond the context of real estate development. We will therefore consider a more general adjustment cost specification in this section. In particular, we will assume that  $K(x) = (x - 1 + \Delta)^\phi$  where  $\Delta$  parameterizes the magnitude of the fixed cost of adjusting. If  $\Delta$  is close to zero, then the fixed cost of adjusting is small. If  $\Delta = 1$ , then adjusting capacity entails the loss of all existing capacity and we recover the standard real estate specification.

## 5.1 Time Between Capacity Adjustments

Investment is much lumpier at the individual project level than at the aggregate level, characterized by periods of intense activity between which very little investment occurs (Doms and Dunne (1998)). We can study this “lumpiness” in investment in our setting in more detail by considering the expected time between successive investments in a project, and the variance of this timing.

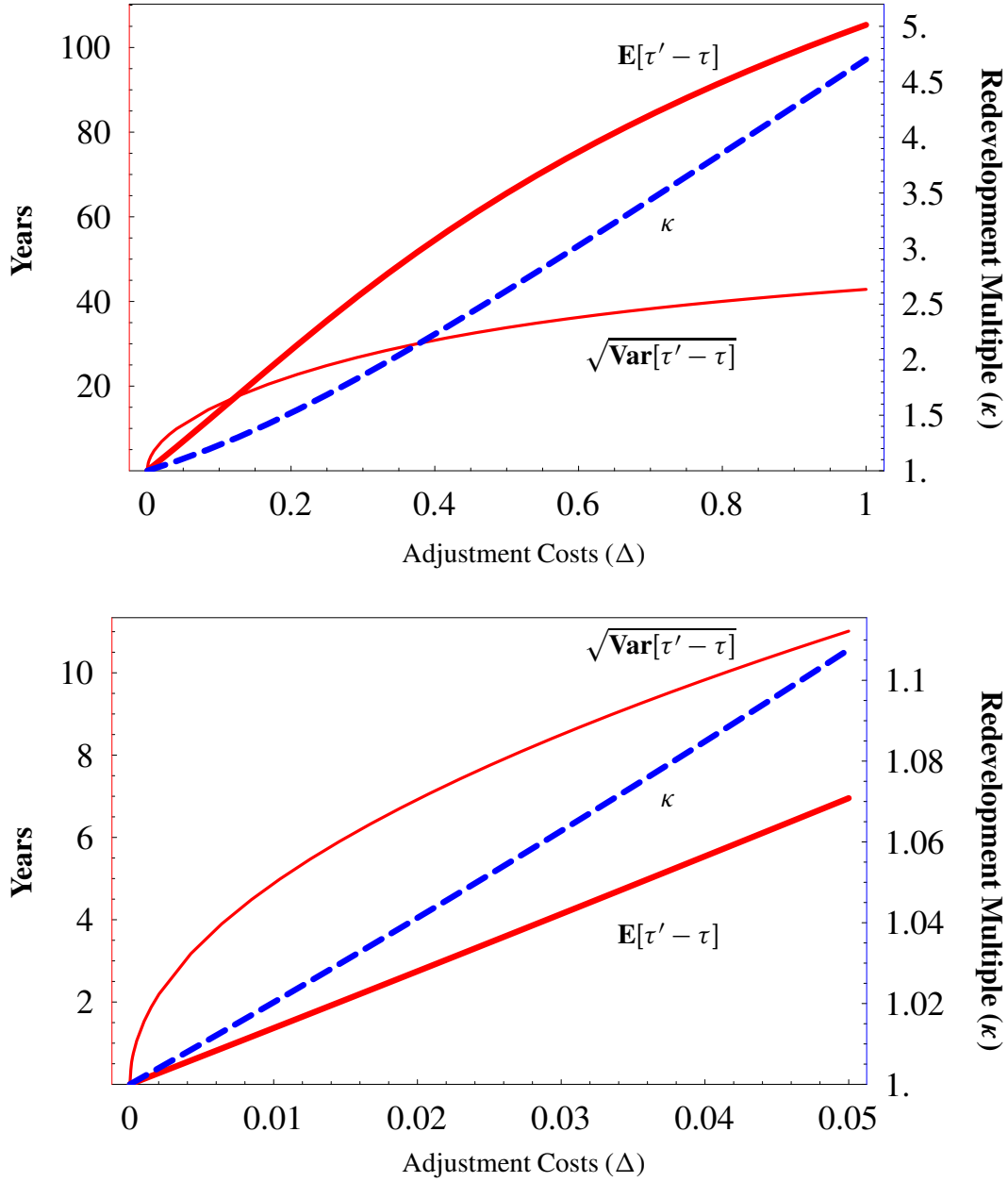
Let  $\tau$  and  $\tau'$  be the stopping time for successive developments. Using standard facts about Brownian processes, together with  $\zeta_{\tau, \tau'} = p_{\tau, \tau'}^{1/\Psi} = \kappa^{(\phi-1)/\Psi}$  and  $\Psi = (\phi-1)\omega/\phi$ , the expectation and variance of time between capacity adjustments at a site are given by

$$\mathbf{E}[\tau' - \tau] = \frac{\frac{\phi}{\omega} \ln \kappa}{\mu_\zeta - \frac{\sigma_\zeta^2}{2}} \quad (68)$$

$$\mathbf{Var}[\tau' - \tau] = \frac{\mathbf{E}[\tau' - \tau]}{\left(\frac{\mu_\zeta}{\sigma_\zeta} - \frac{\sigma_\zeta}{2}\right)^2} \quad (69)$$

where  $\mu_\zeta = \mathbf{1}'\mathbf{M} + \frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{2} + \phi\delta$  and  $\sigma_\zeta^2 = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$ .

Figure 4 shows the expectation and standard deviation of the time between successive capacity adjustments (bold and thin solid curves (red), left hand scale), and the magnitude of these adjustments (dashed curve (blue), right hand scale), as a function of the adjustment costs  $\Delta$ , using our standard parameterization for all other variables. Even small adjustment costs lead to lumpy investment, with firms adjusting capacity infrequently. In the figure adjustment costs of five percent lead developers to increase capacity only every seven years, on average, with a standard deviation of 11 years, and to increase capacity by 10.8 percent when they do. With our standard real estate parameterization of  $\Delta = 1$ , *i.e.*, when redevelopment entails abandonment of assets-in-place, we see



**Figure 4: Expectation and Standard Deviation of Time Between Developments**

Expectation and standard deviation of time between developments (solid bold and thin curves, respectively; left hand scale), and magnitude of the capacity adjustment (dashed curve; right hand scale), as a function of  $\Delta$ , the magnitude of the adjustment costs. The lower panel shows a blow-up of the lower left corner of the top panel. Agents' preferences are given by  $\lambda = 0.02$ ,  $\gamma_c = 1.5$ , and  $a = 2/3$ , the cost-to-scale of development is  $\phi = 1.5$ , the parameters for the stochastic processes are  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ , and housing capital depreciates at  $\delta = 0.01$ .

redevelopment on average every 105 years with a standard deviation of 43 years, with new capacity 4.7 times as large as that which it replaces.

These delays are even more pronounced, and produced by even smaller adjustment costs, if the cost of development is closer to linear. With a cost-to-scale parameter of  $\phi = 1.1$ , for example, adjustment costs of only one percent are required to induce firms to make 10.5 percent adjustments, on average every six years, with a standard deviation of 12 years. With this  $\phi$  and  $\Delta = 1$  sites are redeveloped on average every 186 years, with a standard deviation of 68 years, with new capacity twenty times as large as that which it replaces.

## 5.2 Heterogeneity

In the competitive equilibrium, a developer optimally redevelops a project to  $\kappa$  times its existing capacity. The optimal timing of this development also depends fundamentally, through the dependence of  $p^*$ , on  $\kappa$  (given in equation (39)). The parameter  $\kappa$ , essentially a measure of heterogeneity in the economy, is not arbitrary, but a fixed constant that depends on the economy's primitives.

Again assuming  $K(x) = (x - 1 + \Delta)^\phi$ ,  $\kappa$  is the solution in  $q > 1$  to

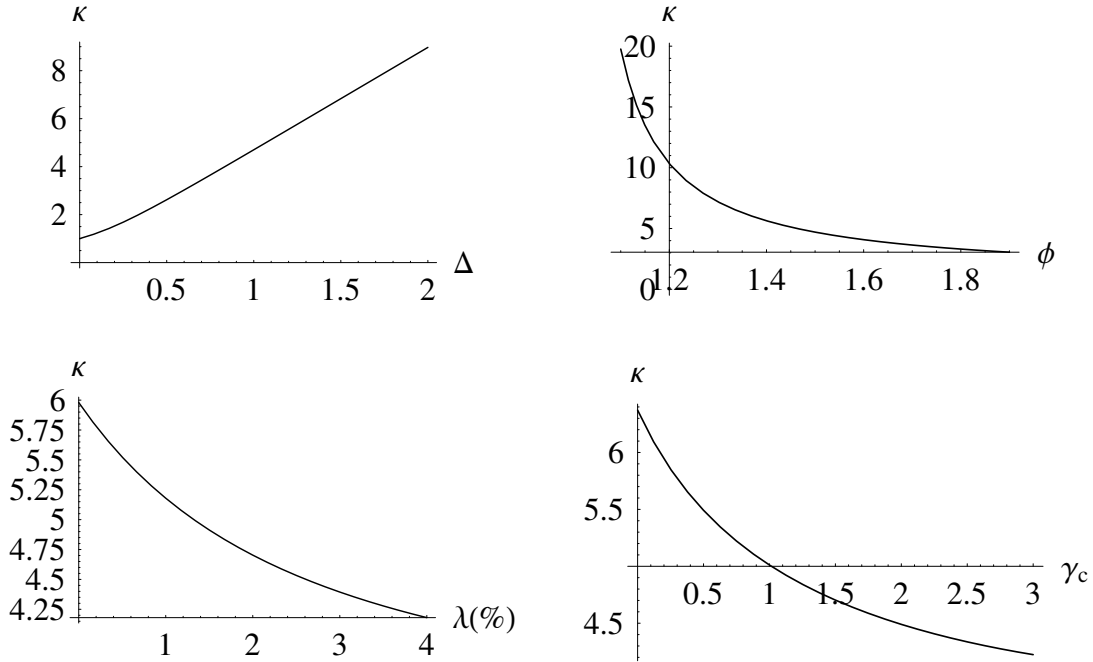
$$\frac{1 + \frac{\Delta}{q-1}}{1 + \frac{\Delta}{q^{(\eta-1)(\phi-1)} - 1}} = \frac{(\eta-1)\phi}{\eta}. \quad (70)$$

This defines  $\kappa$  uniquely, because the left hand side is monotonically decreasing for  $q > 1$ , taking the limits one and  $(\eta - 1)(\phi - 1)$  as  $q$  becomes large and approaches one from above, respectively, and the right hand side lies in this interval. It also implies, as expected, that  $\partial\kappa/\partial\Delta < 0$  (*i.e.*, firms are willing to make smaller capital stock adjustments when adjustment costs are smaller), because the partial derivative of the left hand side with respect to  $\Delta$  is strictly positive.

Figure 5 depicts the degree of heterogeneity in the economy, as functions of the primitives to which it is most sensitive: the magnitude of the adjustment costs ( $\Delta$ ) and the cost-to-scale of adding new capacity ( $\phi$ ), and the impatience and risk aversion of the representative consumer ( $\lambda$  and  $\gamma_C$ ). The figure shows that firms make larger, less frequent adjustments when the fixed cost of adjustment is high (upper left), and smaller adjustments when the cost-to-scale is large (upper right). Firms also make smaller, more frequent adjustments when the representative consumer is impatient (bottom left), or has a strong motive to smooth consumption inter temporally (bottom right).

Importantly, we find a dead weight loss associated with competition, as the competitive equi-





**Figure 5: Heterogeneity (Development Multiple ( $\kappa$ ))**

The ratio of the largest building in the economy to the smallest. Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_c = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

librium yields “too little” heterogeneity. A social planner would utilize the non-housing capital stock more efficiently when developing real estate, by choosing the development multiple  $\kappa^*$  that minimizes the net unit cost of new capacity,

$$\kappa^* = \arg \min_{q > 1} \left\{ \frac{K(q) (\ln q)^{\phi-1}}{(q-1)^\phi} \right\}, \quad (71)$$

which implies that  $\kappa^*$  is the solution in  $q > 1$  to

$$\frac{K'(q)}{K(q)} = \frac{\phi}{q-1} - \frac{\phi-1}{q \ln q}. \quad (72)$$

This implies more heterogeneity than equation (35), the defining equation for  $\kappa$ , which solves the competitive developers' investment problem. In the competitive equilibrium developers do not internalize the negative price externality of their own new capacity, and thus add new capacity “too

soon,” on a scale that is too small to efficiently utilize the resources they employ. This has real economic consequences, as the inefficient use of non-housing capital implies more resources are expended than is necessary for any given increase in the housing capital stock. The associated dead weight loss reduces the average growth rate of both stocks, so the competitive equilibrium growth path diverges from the optimal growth path.

While equations (71) and (72) were derived under an explicit functional assumption regarding the cost of new capacity, the intuition that competitive pressures lead developers to divert resources to the housing sector inefficiently is quite general. Developers internalize the loss from adjustment associated with their own development decision, but do not internalize the marginal impact their investment decision has on the adjustment losses incurred by future developers. By building on a larger scale, a developer would defer other developers’ investment, and thereby defer the deadweight loss associated with this adjustment. Because developers do not internalize the full marginal benefits of developing more intensely, they invest on too small a scale, resulting in a welfare loss.

## 6 Economic Growth

We will now turn our attention to the role equilibrium investment behavior plays in generating a business cycle, in which the expected rate of consumption growth varies with economic conditions. That is, we will explicitly consider the impact nonseparable preferences and the irreversibility constraint have on real consumption growth.

This forces us to make an explicit assumption regarding the “appropriate” numeraire.<sup>12</sup> The obvious choice is appropriately aggregated consumption, *i.e.*, to measure consumption in the same units that denominate the aggregate bundle  $C_t = Y_t^a Q_t^{1-a}$ . In this numeraire consumption is unambiguously related to consumers’ quality of life.<sup>13</sup>

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<sup>12</sup> Up until now we have been able to avoid conclusions that depend on the choice of numeraire. However, when considering growth of any sort the choice of numeraire has a direct material impact on the results, because relative prices change over time. The role numeraire plays can be illustrated simply using “interest rates” as an example. The interest rate is the yield on a default-free bond. In a multi-good economy, however, the “risk-free” asset in the numeraire of the  $i^{\text{th}}$  consumption good, a default-free bond that pays one unit of good  $i$   $T$  in the future, involves price speculation from the point of view of any other arbitrary numeraire,

$$\mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^i \right] = \mathbf{E}_t \left[ e^{-\lambda T} \left( \frac{U_j(C_{t+T})}{U_j(C_t)} \right) \left( \frac{U_i(C_{t+T})/U_j(C_{t+T})}{U_i(C_t)/U_j(C_t)} \right) \right] = \mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^j P_{t,T}^{i/j} \right]$$

where  $P_{t,T}^{i/j}$  is the price in good  $j$  of a unit of good  $i$  at time- $t + T$ , relative to the price at time- $t$ . Similarly, the expected rate of consumption growth depends on what numeraire we use to measure consumption.

<sup>13</sup> This numeraire also results if we assume that 1) consumption transactions are mediated, due to a cash in advance

## 6.1 Consumption Growth

The *expected* average rate of consumption growth is  $\frac{1}{T} \ln \mathbf{E}_t [C_{t,T}]$ . Explicitly evaluating using  $C_{t,T} = e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b$  where  $b = \frac{1-a}{(\phi-1)\eta}$  yields the following proposition.

**Proposition 6.1.** *The expected growth rate of real consumption over the next time  $T$  is given by*

$$\mu_t^C(T) = \mu_C + H\left(\frac{z_t^*}{z_t}, a\beta_{x,\xi}, b, T\right) \quad (73)$$

where  $\mu_C \equiv a(\mu_X + (a-1)\sigma_X^2/2) - (1-a)\delta$  is the drift in aggregate consumption  $C_t$  absent the diversion of new capital into the housing sector;  $b = (1-a)/(\phi-1)\eta > 0$ , and

$$H(x, u, v, T) = \frac{1}{T} \ln\left(F(x, u, v, T) + e^{\Lambda(u,v)T} G(x, u, v, T)\right) \quad (74)$$

where

$$F(x, u, v, T) = N\left(\frac{\ln x}{\sigma_\xi \sqrt{T}} - c\sqrt{T}\right) - \theta x^{2c/\sigma_\xi} N\left(\frac{-\ln x}{\sigma_\xi \sqrt{T}} - c\sqrt{T}\right) \quad (75)$$

$$G(x, u, v, T) = (1+\theta)x^{-v} N\left(\frac{-\ln x}{\sigma_\xi \sqrt{T}} + (c+v\sigma_\xi)\sqrt{T}\right) \quad (76)$$

$$\Lambda(u, v) = v\left(\mu_\xi + (v+2u-1)\frac{\sigma_\xi^2}{2}\right) \quad (77)$$

where  $N(\cdot)$  denotes the cumulative normal,  $\mu_\xi = \mathbf{1}'\mathbf{M} + \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}/2 + \phi\delta$ ,  $\sigma_\xi^2 = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$ ,  $c \equiv \left(\frac{\mu_\xi}{\sigma_\xi} - \frac{\sigma_\xi}{2}\right) + u\sigma_\xi$  and  $\theta \equiv \frac{v\sigma_\xi}{2c+v\sigma_\xi}$ .

In the previous proposition, the first term of equation (73) is the expected growth in consumption absent the conversion technology, resulting from the natural growth of non-housing capital and the depreciation of housing capital. The second term quantifies the impact of the conversion technology, *i.e.*, the effect of expected diversions of capital from the non-housing sector to the housing sector on consumption growth.

Proposition 6.1 also gives, as a corollary, the volatility term structure of expected consumption growth. The expected growth in consumption at any given horizon  $T$  is sensitive to both changes in

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constraint on the purchase of consumption goods, in nominal “dollars” (fiat money), and 2) we adjust prices for inflation using a chained Fisher index, like that used by the U.S. Department of Commerce’s Bureau of Economic Analysis when compiling the National Income and Product Accounts. For details please see appendix C.

$Y_t$  and  $c_t$ , so  $T$ -growth rate volatility is given by

$$\sigma_\mu(T) = \sqrt{\left( \frac{d\mu_t^C(T)}{d \ln X_t}, \frac{d\mu_t^C(T)}{-d \ln c_t} \right) \Sigma \left( \frac{d\mu_t^C(T)}{d \ln X_t}, \frac{d\mu_t^C(T)}{-d \ln c_t} \right)'} \quad (78)$$

This, taken together with the result of Proposition 7.1 and the fact that  $\frac{d \ln z}{d \ln X} = \frac{d \ln z}{-d \ln c} = 1$ , yields the following result.

**Corollary 6.1.** *The volatility of  $T$ -ahead expected real consumption growth is given by*

$$\sigma_\mu(T) = \frac{z^*}{z_t} H' \left( \frac{z^*}{z_t}, a\beta_{x,\xi}, b, T \right) \sigma_\xi. \quad (79)$$

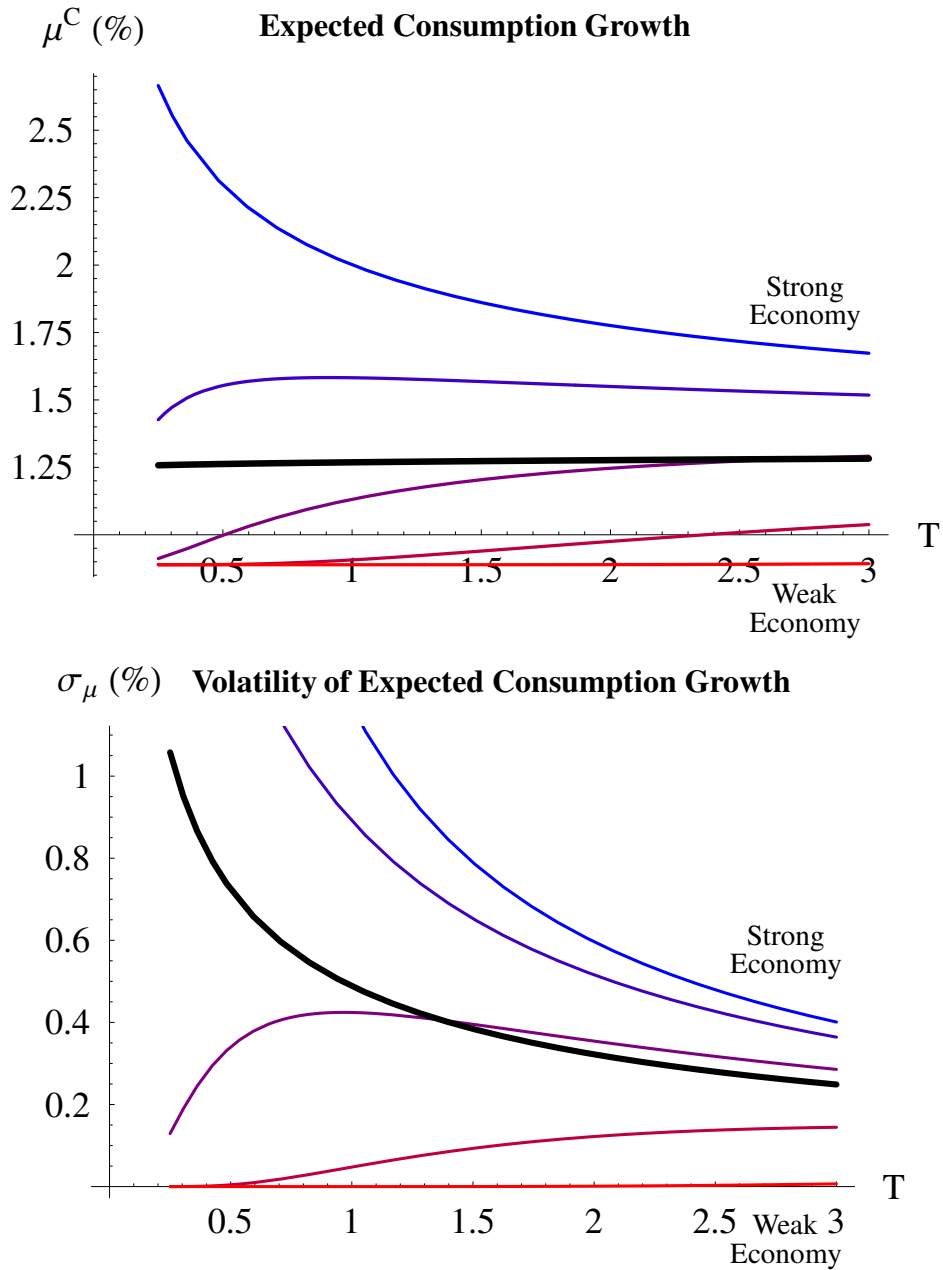
where  $H'(x, a, b, T) = \frac{d}{dx} H(x, a, b, T)$ .<sup>14</sup>

Figure 6 plots the term structure of expected consumption growth (top), and the volatility of expected consumption growth (bottom), out to three years. These are depicted for five different sets of economic conditions, from strong (blue) to weak (red), corresponding to the business cycle variable  $z$  at the 95%, 75%, 50%, 25% and 5% levels of the stationary distribution. Both the expected growth rate and the volatility in this expectation are increasing in the strength of the economy. The bold lines show the growth rates and volatilities unconditionally.

In strong economic conditions the expected growth rate of aggregate consumption is high. At these times the irreversibility constraint does not bind, and capital flows freely to its most productive use, yielding welfare improving adjustments in the composition of consumption. This term structure is downward sloping, reflecting the fact that the investment constraint is likely to bind at some time in the future. In weak economic conditions, when the irreversibility constraint binds, the expected rate of consumption growth is lower. In depressed economic conditions, especially over short horizons when investment will almost certainly not occur, the expected growth rate is equal to what it would be if there simply were no conversion technology. At these times the term structure is upward sloping, reflecting the fact that the investment constraint might not bind at some point in the future. In the long run expected consumption growth converges to its unconditional long-run mean, given in the following corollary.

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<sup>14</sup>  $H'$  has a straightforward analytic expression, which is not included here as it is complicated beyond simple economic interpretation.



**Figure 6: Expected Consumption Growth and Expected Consumption Growth Volatility**

The term structures of expected consumption growth (top) and expected consumption growth volatility (bottom), unconditional (bold lines) and conditional on the business cycle proxy at the 5%, 25%, 50%, 75%, and 95% levels (weakest economy to strongest from red to blue (bottom to top in both figures)). Parameters are  $K(q) = \bar{q}^{3/2}$ ,  $\delta = 0.01$ ,  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ,  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

**Corollary 6.2.** *The long run average rate of real consumption growth,  $\mu_\infty^C = \lim_{T \rightarrow \infty} \mu_t^C(T)$ , is given by*

$$\mu_\infty^C = \mathbf{v}'\mathbf{M} + \frac{\mathbf{v}'\boldsymbol{\Sigma}\mathbf{v}}{2} - (1-a) \left(1 - \frac{\omega}{\Psi\eta}\right) \delta \quad (80)$$

where  $\mathbf{v} = a\mathbf{e}_X + b\mathbf{1}$ , which reduces, if  $\mu_c = \sigma_c = 0$ , to

$$\mu_\infty^C = (a+b) \left( \mu_X + (a+b-1) \frac{\sigma_X^2}{2} \right) - (1-a) \left(1 - \frac{\omega}{\Psi\eta}\right) \delta. \quad (81)$$

The term structure of expected consumption growth rate volatility is relatively flat in weak economic conditions, because investment remains unlikely even after positive shocks. Consequently, while positive shocks increase consumption, they do not increase the expected rate of consumption growth going forward, especially at shorter horizons. In stronger economic conditions the likelihood that the investment constraint is relaxed at short horizons is sensitive to the state of the economy. This results in more volatility in the expected rate of near term consumption growth, generating a downward sloping term structure of expected consumption growth rate volatility.

## 7 Asset Pricing Implications

In the model presented here firms' optimal microeconomic behavior has important macroeconomic implications. In section 4 we determined the optimal equilibrium investment strategy of firms, and doing so explicitly determined the equilibrium pricing kernel. This allows us to compute macro asset pricing fundamentals, providing insight into how firms' decisions impact the term structures of real interest rates and interest rate volatility, the term structure of consumption risk premia, and the term structures of forward prices and forward price volatilities, as well as how each of these term structures varies over the business cycle. In section 8 we will also consider the effects of firms' behavior on the real assets in the economy, determining the expected rates of return on non-housing capital and buildings, and how these vary over the business cycle.

### 7.1 Risk-Free Rate

The real  $T$ -rate is given by  $r_t^f(T) = -\frac{1}{T} \ln \mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C \right]$ , where  $\xi_{t,T}^C = C_{t,T}^{-\gamma_C}$ . Aggregate consumption, in terms of  $X$  and  $\zeta$ , evolves as  $C_{t,T} = e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b$  where  $b = (1 -$

$a)/(\phi - 1)\eta > 0$ . Note that consumption, in the numeraire of aggregate consumption, is less volatile than non-housing consumption, because the elasticity of aggregate consumption with respect to non-housing consumption is  $a < 1$ . Aggregate consumption also grows faster at the investment boundary, because  $b > 0$  implying the development of new housing is welfare improving. The yield curve will therefore slope up in recessions, but down in expansions. Substituting for aggregate consumption in the definition of  $r_t^f(T)$  and evaluating along the lines used in proving proposition 6.1 then yields the following proposition.

**Proposition 7.1.** *The risk-free  $T$ -rate is given by*

$$r_t^f(T) = \lambda + \gamma_C \left( \mu_C - (\gamma_C + 1) \frac{\sigma_C^2}{2} \right) - H \left( \frac{z_t^*}{z_t}, -a\gamma_C \beta_{X,\xi}, -b\gamma_C, T \right) \quad (82)$$

where  $\mu_C = a(\mu_X + (a - 1)\sigma_X^2/2) - (1 - a)\delta$  and  $\sigma_C = a\sigma_X$  are the growth and volatility of aggregate consumption absent development of new housing capacity,  $b = (1 - a)/(\phi - 1)\eta > 0$ , and  $H$  is given in Proposition 6.1.

In equation (82), the first term quantifies consumers' impatience, and the second quantifies the effects of "natural" consumption growth, due to natural growth in the stock of non-housing capital and natural depreciation of housing capital, on the marginal rate of substitution for aggregate consumption. The third term, which depends on the "business cycle" variable  $z_t/z_t^*$ , quantifies the impact of expected future diversions of non-housing resources into the housing sector.

Proposition 7.1 also gives, as a corollary, the term structure of real interest rate volatility. The interest rate at any given horizon  $T$  is sensitive to both changes in  $Y_t$  and  $c_t$ , so  $T$ -rate volatility is given by  $\sigma_r^i(T) = \sqrt{\left( \frac{dr_t^f(T)}{d \ln X_t}, \frac{dr_t^f(T)}{-d \ln c_t} \right) \Sigma \left( \frac{dr_t^f(T)}{d \ln X_t}, \frac{dr_t^f(T)}{-d \ln c_t} \right)'}$ , which taken with the result of Proposition 7.1 yields the following result.

**Corollary 7.1.** *The volatility of the risk-free  $T$ -rate is given by*

$$\sigma_r(T) = \frac{z_t^*}{z_t} H' \left( \frac{z_t^*}{z_t}, -a\gamma_C \beta_{X,\xi}, -b\gamma_C, T \right) \sigma_\xi \quad (83)$$

where  $H'(x, a, b, T) = \frac{d}{dx} H(x, a, b, T)$ .

Figure 7 plots the term structure of interest rates (top) and interest rate volatilities (bottom) out to three years. These are depicted for five different set of economic conditions, from strong (blue)

to weak (red), corresponding to the business cycle variable  $z$  at the 95%, 75%, 50%, 25% and 5% levels of the stationary distribution. Both interest rates and interest rate volatilities are increasing in the strength of the economy, properties inherited from the expected rate of consumption growth. The bold lines show the unconditional yield curve and the unconditional term structure of interest rate volatility.

In weak economic conditions there is some horizon over which firms are nearly certain not to invest in new housing capital, and the risk-free rate agrees with the rate in an economy without the conversion technology. The weaker the economy, the longer the horizon over which investment is unlikely, resulting in a low, flat term structure in severe recession. As the economy strengthens, and investment becomes more likely, the interest rate at these horizons begins to increase, resulting in a low, upward sloping term structure. As the economy strengthens further, and investment becomes more likely at shorter horizons, the short end begins to rise faster than the long end, resulting in a high, flat term structure. Finally, as the economy strengthens still further, the short end rises dramatically, resulting in a high, downward sloping term structure in expansions. Taken together, these imply a term structure slope hump-shaped in the strength of the economy. In all cases the rates asymptote to the “long rate,” given in the following corollary to Proposition 7.1.

**Corollary 7.2.** *The “long rate”  $r_\infty^f \equiv \lim_{T \rightarrow \infty} r_{t,T}^f$  is given by*

$$r_\infty^f = \lambda + \gamma_C \left( \mu_C - (\gamma_C + 1) \frac{\sigma_C^2}{2} \right) + \frac{\rho}{\Psi \eta} \gamma_C \delta + b \gamma_C \left( \mathbf{1}' \mathbf{M} - (b + 2a\beta_{X,\xi}) \gamma_C \frac{\mathbf{1}' \Sigma \mathbf{1}}{2} \right), \quad (84)$$

which reduces, if  $\mu_C = \sigma_C = 0$ , to

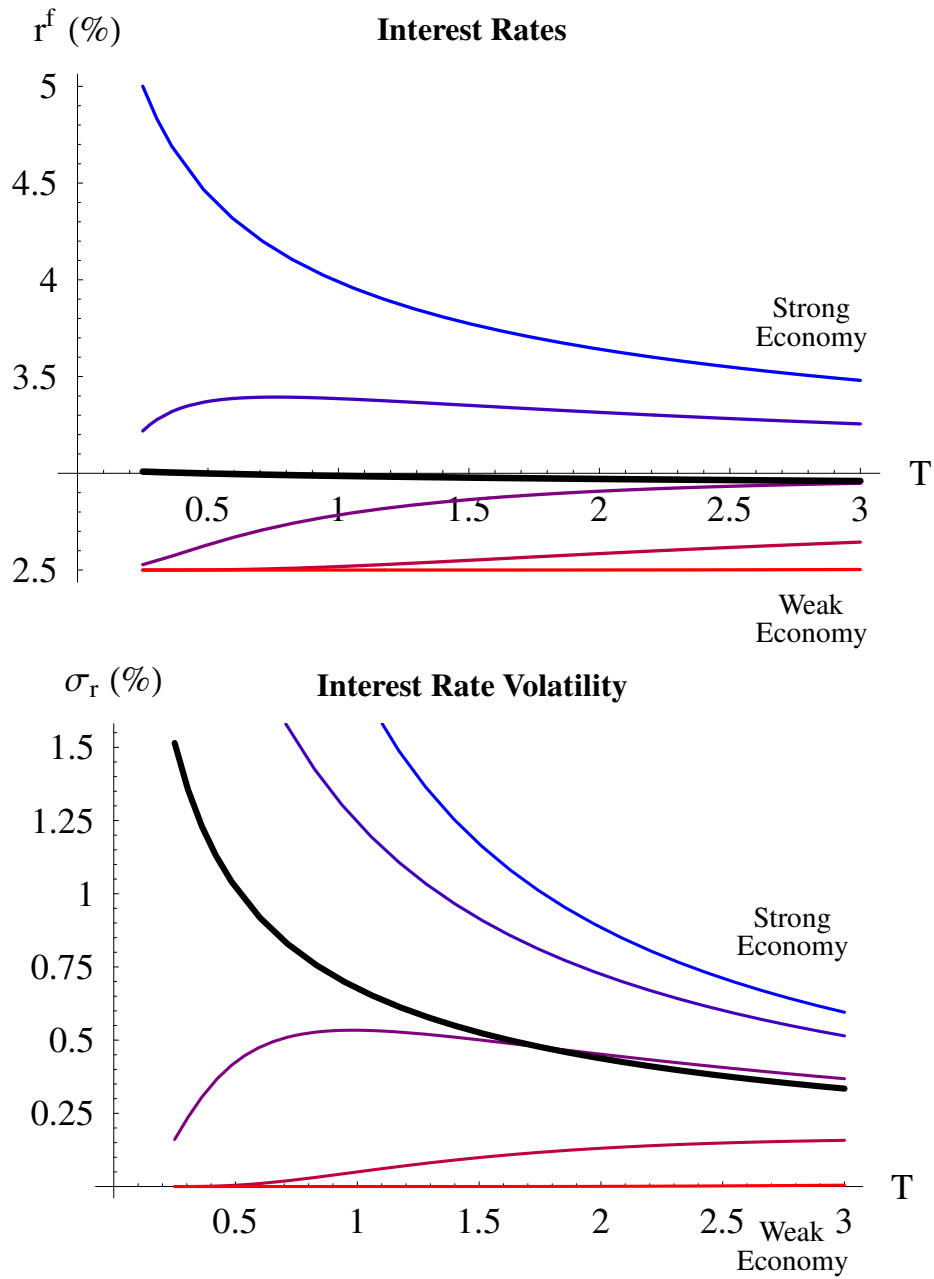
$$r_\infty^f = \lambda + \gamma_C \left( \mu_\infty^C - (\gamma_C + 1) \frac{\sigma_C^{*2}}{2} \right) \quad (85)$$

where  $\mu_\infty^C$  is the long run average rate of consumption growth, given in corollary 6.2, and  $\sigma_C^* = (a + b) \sigma_X$  is the annualized variance of long run consumption growth.

## 7.2 The Term Structure of Consumption Risk Premia

To study consumption premia at different horizons we will analyze “consumption bonds.” These mature on a fixed date, and pay a dividend that depends on the aggregate consumption basket on





**Figure 7: Term Structures of Interest Rates and Interest Rate Volatility**

Term structure of interest rates (top) and interest rate volatility (bottom), unconditional (bold line) and conditional on the business cycle proxy at the 5%, 25%, 50%, 75%, and 95% levels (strong economy (top, blue), to weakest (bottom, red)). Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

the date of maturity. The bond written at  $t$  maturing at  $t + T$  delivers  $Y_{t,T}$  units of non-housing goods and  $Q_{t,T}$  units of housing services, *i.e.*, a quantity of each good in proportion to the ratio of the corresponding aggregate at maturity to the date of issue.

The expected return is the log of the expected pay off / issue price ratio, all divided by the time to maturity, *i.e.*,

$$r_t^c(T) = \frac{1}{T} \left( \ln \mathbf{E}_t [C_{t,T}] - \ln \mathbf{E}_t \left[ e^{-\lambda T} C_{t,T}^{1-\gamma_C} \right] \right). \quad (86)$$

Evaluating the right hand side of the previous equation yields the following proposition.

**Proposition 7.2.** *The expected yield on a consumption bond with  $T$ -to-maturity is*

$$\begin{aligned} r_t^c(T) = & \lambda + \gamma_C \left( \mu_C + (1 - \gamma_C) \frac{\sigma_C^2}{2} \right) + H \left( \frac{z^*}{z_t}, a\beta_{X,\xi}, b, T \right) \\ & - H \left( \frac{z^*}{z_t}, a(1 - \gamma_C) \beta_{X,\xi}, b(1 - \gamma_C), T \right). \end{aligned} \quad (87)$$

where  $\mu_C = a(\mu_X + (a - 1)\sigma_X^2/2) - (1 - a)\delta$  and  $\sigma_C = a\sigma_X$  are the growth and volatility of aggregate consumption absent development of new housing capacity,  $b = (1 - a)/(\phi - 1)\eta > 0$ , and  $H$  is given in Proposition 6.1.

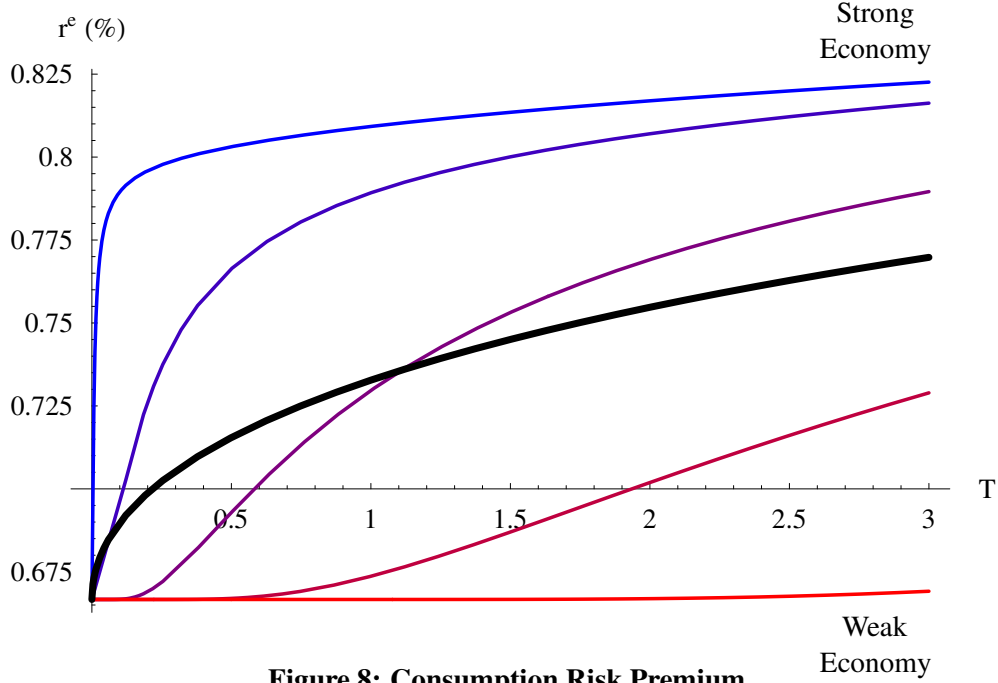
Propositions 7.1 and 7.2 also provide, as an immediate corollary, the term structure of consumption risk premia.

**Corollary 7.3.** *The term structure of expected excess returns  $r_t^e(T) = r_t^c(T) - r_t^f(T)$  is given by*

$$\begin{aligned} r_t^e(T) = & \gamma_C \sigma_C^2 + H \left( \frac{z^*}{z_t}, -a\gamma_C \beta_{X,\xi}, -b\gamma_C, T \right) \\ & + H \left( \frac{z^*}{z_t}, a\beta_{X,\xi}, b, T \right) - H \left( \frac{z^*}{z_t}, a(1 - \gamma_C) \beta_{X,\xi}, b(1 - \gamma_C), T \right). \end{aligned} \quad (88)$$

Figure 8 shows the term structure of consumption risk premia, *i.e.*, the expected excess returns to consumption bonds, out to three years, for the same levels of the business cycle proxy used in Figure 7, the 5%, 25%, 50%, 75% and 95% levels of the stationary distribution (weak economic conditions, bottom (red); strong economic conditions, top (blue)), and unconditionally (bold curve). The consumption risk premium at any horizon is pro-cyclical, increasing with the strength of the economy. In general, the premium agrees at the extreme short end with the consumption risk premium in an economy in which no conversion technology exists, as there are some horizons over

which firms are nearly certain not to invest in new housing capital. It converges asymptotically to the long horizon consumption risk premium, given in the following corollary.



**Figure 8: Consumption Risk Premium**

Term structure of expected excess returns to consumption bonds, unconditional (bold line) and conditional on the business cycle proxy at the 5%, 25%, 50%, 75%, and 95% levels (weakest economy (bottom, red) to strongest (top, blue)). Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

**Corollary 7.4.** The “long rate” on consumption bonds  $r_\infty^c \equiv \lim_{T \rightarrow \infty} r_t^c(T)$  is given by

$$r_\infty^c = r_\infty^f + \gamma_C (\mathbf{a}\mathbf{e}_X + \mathbf{b}\mathbf{1})' \Sigma (\mathbf{a}\mathbf{e}_X + \mathbf{b}\mathbf{1}) \quad (89)$$

which reduces, if  $\mu_c = \sigma_c = 0$ , to

$$r_\infty^c = r_\infty^f + \gamma_C \sigma_C^{*2} \quad (90)$$

where  $\sigma_C^* = (a + b) \sigma_X$ .

### 7.3 Forward Prices of the Two Consumption Goods

The  $T$ -ahead forward price for delivery of the non-housing consumption good, or housing services, is the risk-adjusted expected future spot price, so satisfies  $\mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C (P_{t+T}^i - F_t^i(T)) \right] = 0$  where the superscript  $i \in (Y, Q)$  denotes the consumption good. Forward prices are therefore given by

$$F_t^i(T) = \left( \frac{\mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C P_{t,T}^i \right]}{\mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C \right]} \right) P_t^i. \quad (91)$$

Switching numeraires in the numerator of the previous equation and taking logs yields

$$\frac{F_t^i(T)}{P_t^i} = \exp \left( \left[ r_t^f(T) - r_t^i(T) \right] T \right) \quad (92)$$

where  $r_t^i(T) \equiv \frac{1}{T} \ln \mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^i \right]$  is the “interest rate,” using good  $i$  as numeraire, on a risk-free bond that provides one unit of good  $i$  time- $T$  in the future. That is, forward prices are given by a commodities version of the familiar covered interest parity relation for forward exchange rates. Calculating the “interest rates” in the numeraires of the non-housing consumption good and housing services explicitly then yields the following proposition.

**Proposition 7.3.** *The forward prices for delivery of the non-housing consumption good and housing services time- $T$  in the future are given by*

$$F_t^i(T) = P_t^i \exp \left( \left( r_t^f(T) - r_t^i(T) \right) T \right) \quad (93)$$

where  $r_t^f(T)$  is given in proposition 7.1 and

$$r_t^Y(T) = \lambda + \gamma_Y \left( \mu_X - (\gamma_Y + 1) \frac{\sigma_X^2}{2} \right) + (1 - \gamma_Q) \delta - H \left( \frac{z^*}{z_t}, -\gamma_Y \beta_{X,\xi}, \Omega, T \right) \quad (94)$$

$$r_t^Q(T) = \lambda - (1 - \gamma_Y) \left( \mu_X - \gamma_Y \frac{\sigma_X^2}{2} \right) - \gamma_Q \delta - H \left( \frac{z^*}{z_t}, (1 - \gamma_Y) \beta_{X,\xi}, \Omega + \Psi - 1, T \right). \quad (95)$$

The proposition also yields, as a corollary, the term structures of the forward price volatilities. The  $T$ -ahead delivery price volatility is  $\sigma_F^i(T) = \sqrt{\left( \frac{d \ln F_t^i(T)}{d \ln X_t}, \frac{d \ln F_t^i(T)}{-d \ln c_t} \right) \Sigma \left( \frac{d \ln F_t^i(T)}{d \ln X_t}, \frac{d \ln F_t^i(T)}{-d \ln c_t} \right)}$ , which, taken with the result of Proposition 7.3 and using  $\frac{d \ln P^Y}{d \ln X} = a - 1$ ,  $\frac{d \ln P^Q}{d \ln X} = a$  and

$\frac{d \ln P^Y}{-d \ln c} = \frac{d \ln P^Q}{-d \ln c} = 0$ , yields the following result.

**Corollary 7.5.** *The volatilities of the  $T$ -ahead forward prices of non-housing consumption goods and housing services are given by*

$$\sigma_F^Y(T) = \sqrt{\left( (a-1)\mathbf{e}_X + J^Y\left(\frac{z^*}{z}, T\right)\mathbf{1} \right)' \Sigma \left( (a-1)\mathbf{e}_X + J^Y\left(\frac{z^*}{z}, T\right)\mathbf{1} \right)} \quad (96)$$

$$\sigma_F^Q(T) = \sqrt{\left( a\mathbf{e}_X + J^Q\left(\frac{z^*}{z}, T\right)\mathbf{1} \right)' \Sigma \left( a\mathbf{e}_X + J^Q\left(\frac{z^*}{z}, T\right)\mathbf{1} \right)} \quad (97)$$

where

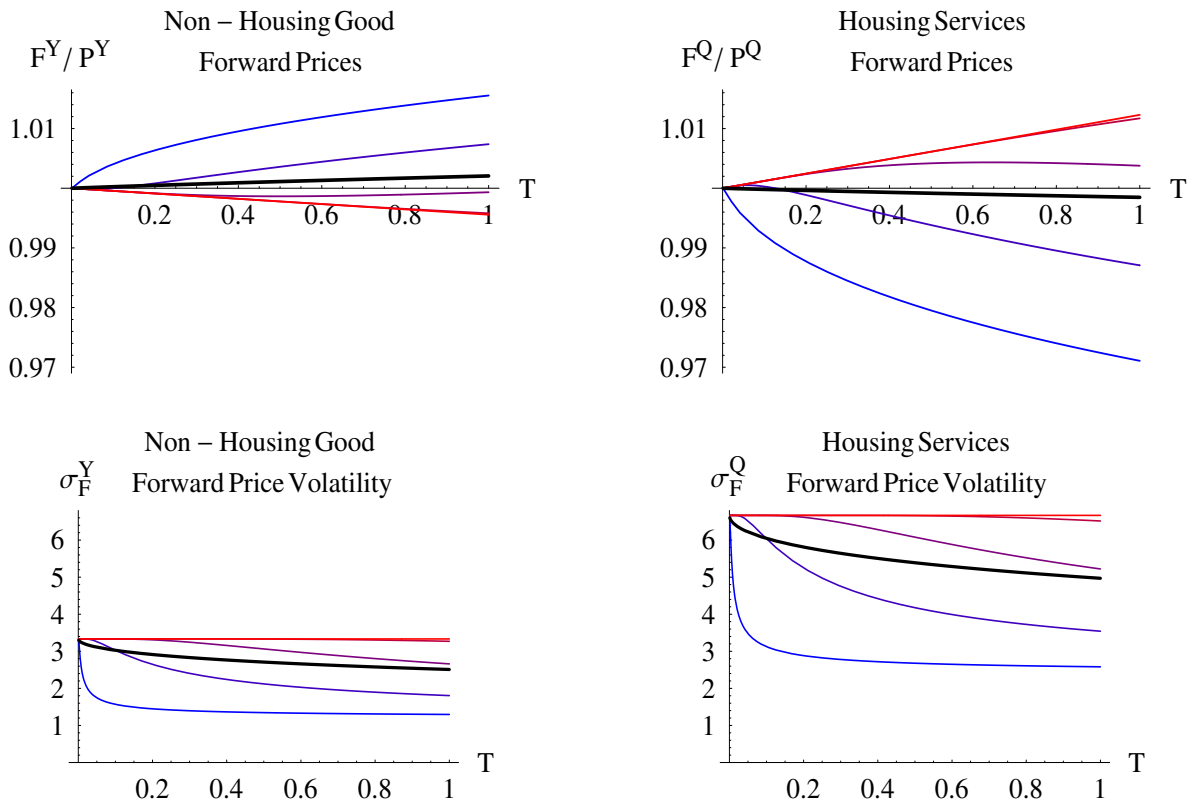
$$\frac{J^Y(x, T)}{xT} = H'(x, -\gamma_Y \beta_{x,\xi}, \Omega, T) - H'(x, -a\gamma_C \beta_{x,\xi}, -b\gamma_C, T) \quad (98)$$

$$\frac{J^Q(x, T)}{xT} = H'(x, (1 - \gamma_Y) \beta_{x,\xi}, \Omega + \Psi - 1, T) - H'(x, -a\gamma_C \beta_{x,\xi}, -b\gamma_C, T) \quad (99)$$

and  $H'(x, a, b, T) = \frac{d}{dx} H(x, a, b, T)$ .

The term structure of forward prices and forward price volatility are given below, in Figure 9. In recessions, when housing consumption is high relative to non-housing consumption, natural growth in the non-housing sector and depreciation in the housing sector both tend to make future non-housing consumption relatively abundant, and thus relatively cheap, lowering its forward price. Conversely, in expansions the expected diversion of resources to the housing sector raises the expected future price of the non-housing goods, by both decreasing expected future non-housing consumption and increasing expected housing consumption, which increases the price for future delivery. Consequently, the forward market for non-housing goods is backwardated in recessions and in contango in expansions. The opposite holds true in the forward market for housing services. In recessions the increasing relative scarcity of housing services generates an upward sloping forward price curve, while in expansions the increasing relative abundance of housing services generates downward sloping forward price curves.

The Samuelson Hypothesis holds in both markets, with forward price volatility increasing as delivery approaches, *i.e.*, the term structures of the forward price volatilities are downward sloping. This reflects the fact that investment in the housing sector, which occurs in response to a high relative price for housing services, reduces the relative price of these services. This generates mean reversion in relative prices in expansions, or at long horizons, when housing supply is more elastic. These



**Figure 9: Forward Prices and Forward Price Volatilities**

Forward price and forward price volatilities (top and bottom, respectively) for non-housing goods and housing services (left and right, respectively), as functions of time-to-delivery. Figures show these unconditionally (bold lines), and conditional on business cycle proxies at the 5%, 25%, 50%, 75%, and 95% levels, weakest to strongest from red to blue (bottom to top in the upper left panel, top to bottom in the other panels). Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

term structures of forward price volatilities slope more steeply downward in stronger economic conditions, so the term structures of forward prices and forward price volatilities correlate negatively in the non-housing good forward market (*i.e.*, the term structure of forward price volatility slopes more steeply downward in contango markets than it does in backwardated markets), but correlate positively in the housing services forward market (*i.e.*, the term structure of forward price volatility slopes more steeply downward in backwardated markets than it does in contango markets).

## 8 Returns to Real Assets

We will now turn our attention to the expected excess returns of the real assets in the economy, and how they vary over the business cycle. In particular, we will consider how much compensation, in the form of expected returns, investors demand for exposure to aggregate uncertainty through positions in non-housing capital and buildings, and how this compensation depends on the state of the economy.

### 8.1 Expected Return to Non-Housing Capital

The total return on non-housing capital (or any asset) consists of two parts: the dividend yield, which includes any “cash flows” to the asset holder, and the capital gain, which accounts for any movements in the asset’s price, including those due to user costs (*e.g.*, depreciation).

The dividend yield is the reciprocal of the price/earnings ratio, which we gave explicitly for non-housing capital in equation (67). The value of non-housing capital is its price/earnings ratio times its earnings, or

$$V^I \left( X, P^Y, \frac{z}{z^*} \right) = X P^Y \left[ \pi + \left( \frac{z}{z^*} \right)^{\Psi\eta + \Omega - 1} \left( \Pi^I - \pi \right) \right] \quad (100)$$

where  $X$  is the capital’s output of the non-housing good and  $P^Y$  is the price of the good. The expected capital gain is the value weighted average of the expected capital gains of the two terms in the previous equation. Using the fact that  $X_{t,s} P_{t,s}^Y = e^{(a-1)\delta s} X_{t,s}^a = C_{t,s}$  on an interval on which no new housing capital is developed, we then have that the expected return on non-housing capital in the numeraire of aggregate consumption, including both the dividend yield and the expected capital gain, is equal to

$$r^I \left( \frac{z_t}{z^*} \right) = \left( \frac{\pi}{\Pi^I \left( \frac{z}{z^*} \right)} \right) (\pi^{-1} + \mu_C) + \left( 1 - \frac{\pi}{\Pi^I \left( \frac{z}{z^*} \right)} \right) \frac{\mathbf{E} \left[ \frac{dC z^{\Psi\eta + \Omega - 1}}{C z^{\Psi\eta + \Omega - 1}} \right]}{dt} \quad (101)$$

where  $\Pi^I \left( \frac{z}{z^*} \right) = \pi + \left( \frac{z}{z^*} \right)^{\Psi\eta + \Omega - 1} (\Pi^I - \pi)$  is the price/earnings ratio for non-housing capital. Evaluating the previous equation explicitly yields the following proposition.

**Proposition 8.1.** *The expected instantaneous total return to non-housing capital is*

$$r^I\left(\frac{z}{z^*}\right) = r_0^c + w\left(\frac{z}{z^*}, 0, \Omega\right)(R - r_0^c) \quad (102)$$

where  $r_0^c \equiv \lim_{T \rightarrow \infty} r_t^c(T) = \lambda + \gamma_C \left( \mu_C + (1 - \gamma_C) \frac{\sigma_C^2}{2} \right)$  is the consumption bond short rate,  $R$ , the expected return to holding business cycle risk, is given by

$$R = \mathbf{u}'\mathbf{M} + \frac{\mathbf{u}'\Sigma\mathbf{u}}{2} + ((\Psi\eta + \Omega - 1)\phi + a - 1)\delta \quad (103)$$

where  $\mathbf{u} \equiv (\Psi\eta + \Omega - 1)\mathbf{1} + a\mathbf{e}_X$ , and the weight on the expected return to business cycle risk is given by  $w(x, y, \alpha)^{-1} \equiv 1 + x^{-(\Psi\eta + \Omega - 1)} \left( \frac{(\Psi\eta + \Omega - 1) - \alpha}{\alpha + (\Psi\eta + \Omega - 1)y} \right)$ .

The expected rate of return given in equation (102) is a weighted average of  $r_0^c$ , the expected return to non-housing capital that would prevail if there were no technology for building new housing, and  $R$ , which represents a return to “business cycle risk,” where the weight on the consumption bond short rate,  $1 - w\left(\frac{z}{z^*}, 0, \Omega\right)$ , is just an explicit parameterization of  $\pi / \Pi^I\left(\frac{z}{z^*}\right)$ . Non-housing capital is exposed to business cycle risk, because the diversion of non-housing capital to the housing sector in expansions results in a relative scarcity of non-housing consumption. This increases the current value of expected future production, and consequently of the capital that produces non-housing goods, especially in strong economic conditions when near-term conversion is more likely.

## 8.2 Expected Return to Buildings

The capitalization rate for buildings, *i.e.*, a building’s dividend yield, was given explicitly in equation (66). The value of a building is its cap-rate times the revenue it produces, or

$$\begin{aligned} V^{II}\left(q, P^Q, \frac{z}{z^*}, \underline{q}\right) & \\ = qP^Q \left[ \pi + \left(\frac{z}{z^*}\right)^{\Psi\eta + \Omega - 1} \left( \left( \Pi^{II} - \pi \right) + \left(\frac{q}{\underline{q}}\right)^{(\eta - 1)(\phi - 1)} \left( \frac{\kappa - 1}{\eta(1 - \kappa^{\eta + \phi - \eta\phi})} \right) \Pi^{II} \right) \right] & \end{aligned} \quad (104)$$

where  $q$  is the quantity of housing services provided by the building and  $P^Q$  is the price of the good. The expected capital gain is the value weighted average of the expected capital gains of the terms of the previous equation. Using the fact that  $q_{t,s} P_{t,s}^Q = e^{(a-1)\delta s} X_{t,s}^a = C_{t,s}$  on an interval on which no new housing capital is developed, we then have that the instantaneous expected return



on the building is given by

$$r^{II}\left(\frac{q}{q}, \frac{z_t}{z^*}\right) = \left(\frac{\pi}{\Pi^{II}\left(\frac{q}{q}, \frac{z_t}{z^*}\right)}\right)(\pi^{-1} + \mu_C) + \left(1 - \frac{\pi}{\Pi^{II}\left(\frac{q}{q}, \frac{z_t}{z^*}\right)}\right) \frac{\mathbf{E}\left[\frac{dCz^{\Psi\eta+\Omega-1}}{Cz^{\Psi\eta+\Omega-1}}\right]}{dt}. \quad (105)$$

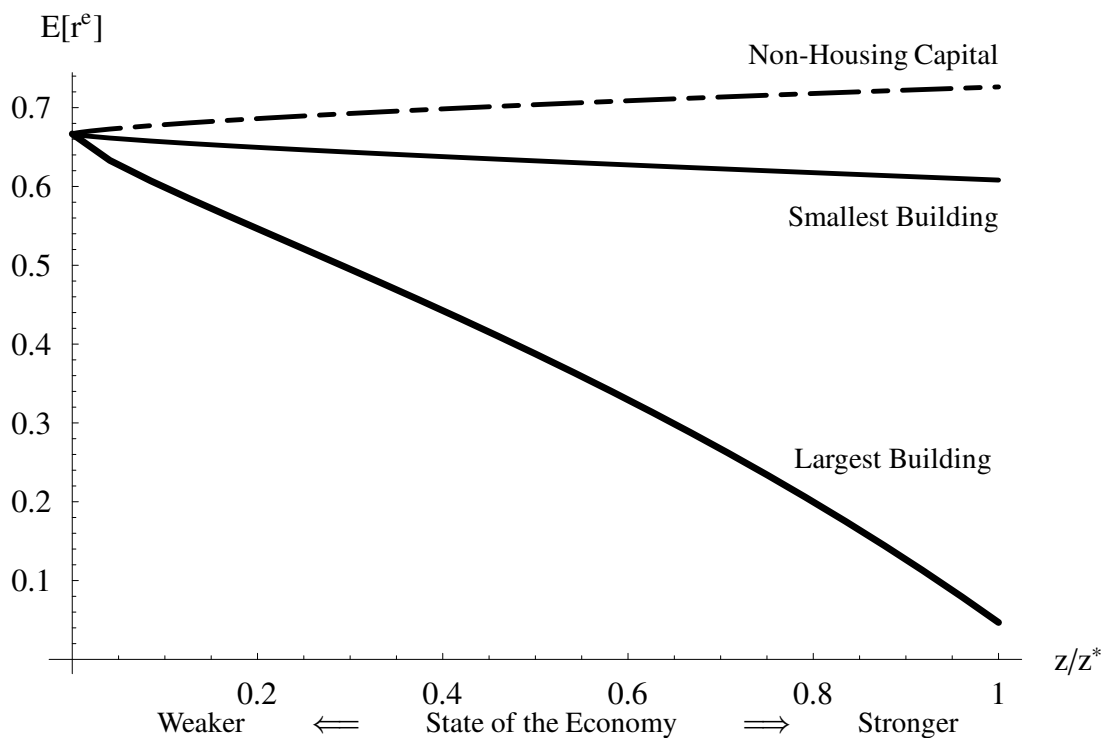
Evaluating the previous equation explicitly yields the following proposition.

**Proposition 8.2.** *The expected instantaneous total return to a building developed to  $q/q$  times the minimum current density is*

$$r^{II}\left(\frac{q}{q}, \frac{z_t}{z^*}\right) = r_0^c + w\left(\frac{z_t}{z^*}, \left(\frac{q}{q}\right)^{(\eta-1)(\phi-1)} \Theta, \Omega - 1 + \Psi\right) (R - r_0^c) \quad (106)$$

where  $\Theta = \frac{\kappa-1}{\eta(1-\kappa\eta+\phi-\eta\phi)}$ , and  $r_0^c$ ,  $R$  and  $w(x, y, \alpha)$  are given in proposition 8.1.

Figure 10 shows the expected excess rates of return on non-housing capital and the smallest and largest buildings in the economy, as functions of the business cycle variable  $z/z^*$ . Buildings carry a positive risk premium, because the value of housing assets-in-place is pro-cyclical, *i.e.*, exposed to consumption growth risk, rising with growth in the output of non-housing goods. This premium is counter-cyclical, however, because the exposure of housing assets-in-place to natural growth in the non-housing capital stock is mitigated, especially in strong economic conditions, by the fact that rising non-housing consumption eventually elicits new housing, reducing the production-share of assets-in-place. This supply effect partially offsets the demand effect: while increasing non-housing good production has the direct effect of increasing the price of housing services, and consequently the revenue from housing assets-in-place, its indirect effect is to elicit new housing supply, which reduces the price of housing services, and consequently the revenues from existing housing assets. As a result, revenues from housing assets-in-place covary less strongly with the pricing kernel over the long run, reducing the return premium, especially in strong economic conditions when the housing supply is relatively elastic. Smaller buildings have higher expected returns than large buildings, because a greater fraction of their value derives from the growth option to build a bigger building in the future. This growth option is an exposure to business cycle risk, which carries a high premium. The expected return on non-housing capital is higher than the expected return to buildings because non-housing capital is, not surprisingly, more exposed to the primary risk factor in the economy, the natural growth of non-housing capital. The magnitude of the premium on non-housing capital in excess of that on buildings is also pro-cyclical. The diversion of non-housing capital to the housing



**Figure 10: Expected Excess Return on Non-Housing Capital and Buildings**

The instantaneous expected rate of return on trees (middle curve), housing assets-in-place (bottom curve) and growth options (top curve), as a function of the business cycle variable  $z = Y/Q^{\phi}c$ . Parameters are, for the conversion technology:  $K(q) = q^{3/2}$  and  $\delta = 0.01$ ; for the agents' preferences:  $\lambda = 0.02$ ,  $\gamma_C = 1.5$ , and  $a = 2/3$ ; and for the technological shocks:  $\mu_X = 0.02$ ,  $\sigma_X = 0.1$ ,  $\mu_c = 0$ ,  $\sigma_c = 0.03$  and  $\rho = 0$ .

sector results in higher marginal utility of non-housing consumption, increasing the covariance of the pricing kernel with the output of non-housing capital in expansions. This increases the premium demanded on non-housing capital in expansions relative to recessions. This, in conjunction with the counter-cyclical premium on buildings, implies the pro-cyclical wedge in the rates of return on non-housing capital and buildings.

## 9 Conclusion

When preferences over housing and other goods and services are nonseparable, investment constraints on altering the aggregate stock of housing have real, first-order economic consequences. Moreover, the asymmetric nature of the constraint, irreversibility that makes disinvestment in bad

times more difficult than investment in good times, means the impact of these investment frictions varies over the business cycle. This generates endogenous time-variation in the expected rate of consumption growth, even when the shocks driving the economy show no such variation, because welfare improving adjustments in the composition of consumption only occur at times when the investment constraint does not bind. This has broad macroeconomic implications. We show that nonseparable preferences and investment irreversibility imply a term structure slope of real interest rates that is hump-shaped in economic strength, an interest rate volatility term structure that is unconditionally downward sloping but conditionally hump-shaped, an upward sloping term structure of consumption risk premia that slopes more steeply in expansions than in recessions, and forward markets that may be either backward dated or in contango, depending on the state of the economy. We also show that the frictions are consistent with both the time-series and cross-section of investment, generating “lumpy” investment at the firm level and heterogeneity across firms.

## A Implied housing service consumption in PST

PST assume standard CRRA over aggregated consumption,

$$U(Y_t, Q_t) = \mathbf{E}_t \left[ \int_0^\infty e^{-\lambda s} \left( \frac{C(Y_{t+s}, Q_{t+s})^{1-\gamma}}{1-\gamma} \right) ds \right] \quad (107)$$

where non-housing consumption and housing services are aggregated assuming constant intratemporal elasticity of substitution  $\epsilon$ ,

$$C(Y_t, Q_t) = \left( Y_t^{\frac{\epsilon-1}{\epsilon}} + w Q_t^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}. \quad (108)$$

In the limit as  $\epsilon \rightarrow 1$  the aggregation is of Cobb-Douglas form. PST's preferred parameters for  $\epsilon$  are 1.05 and 1.25, which are close to Cobb-Douglas.

PST define the non-housing consumption share as

$$\alpha_t = \frac{P_t^Y Y_t}{P_t^Y Y_t + P_t^Q Q_t}, \quad (109)$$

where  $P_t^Y$  and  $P_t^Q$  denote the unit prices of non-housing consumption and housing services. Using the previous definition, and the fact that

$$\frac{P_t^Q}{P_t^Y} = w \left( \frac{Q_t}{Y_t} \right)^{1/\epsilon}, \quad (110)$$

housing consumption at  $t$  can be written as

$$Q_t = w^{\frac{\epsilon}{\epsilon-1}} (\alpha_t^{-1} - 1)^{\frac{\epsilon}{\epsilon-1}} Y_t. \quad (111)$$

The ratio of housing consumption at  $t + s$  relative to  $t$  is, therefore, given by

$$Q_{t,s} = \left( \frac{\alpha_{t+s}^{-1} - 1}{\alpha_t^{-1} - 1} \right)^{\frac{\epsilon}{\epsilon-1}} Y_{t,s}. \quad (112)$$

The drop in housing consumption share  $(1 - \alpha)$  from 19% in the mid-1980s to 17% in 2000 implies, if  $\epsilon = 1.25$ , that the consumption of real housing services fell nearly 50% over the period, and implies, if  $\epsilon = 1.05$ , that the consumption of real housing services fell more than 90%. In the period encompassing the Great Depression the housing consumption share fell from 23%, in the early '30s, to 12%, in the late '40s. If  $\epsilon = 1.25$  this implies real housing consumption was 30 times as high at the start of the sample as at the end of the sample. If  $\epsilon = 1.05$  it implies real housing consumption was 2,000,000 times as high at the start of the sample as at the end of the sample. These numbers, which seem implausible, are required to generate the volatility in "true aggregate consumption" necessary to generate a large equity premium using CRRA preferences.

## B Proofs of Propositions

### Proof of Proposition 4.1

*Proof of the proposition:* The perpetuity factor is given by

$$\begin{aligned}
\pi &\equiv \mathbf{E}_0 \left[ \int_0^\infty e^{-\hat{\lambda}t} X_{0,t}^{1-\gamma_Y} dt \right] \\
&= \mathbf{E}_0 \left[ \int_0^\infty e^{-\hat{\lambda}t} \mathbf{E}_0 \left[ X_{0,t}^{1-\gamma_Y} \right] dt \right] \\
&= \frac{1}{\hat{\lambda} - (1-\gamma_Y) \left( \mu_X - \gamma_Y \frac{\sigma_X^2}{2} \right)}.
\end{aligned} \tag{113}$$

Then, because  $X_{0,t} \zeta_{0,t}^{-\beta_{X,\zeta}}$  is independent of  $X_{0,t}$  for  $\beta_{X,\zeta} = \mathbf{e}'_X \boldsymbol{\beta}$  where  $\boldsymbol{\beta} \equiv \frac{\boldsymbol{\Sigma} \mathbf{1}}{\mathbf{1}' \boldsymbol{\Sigma} \mathbf{1}}$ ,

$$\begin{aligned}
\mathbf{E}_0 \left[ e^{-\hat{\lambda}T} X_{0,T}^{1-\gamma_Y} \right] &= \mathbf{E}_0 \left[ e^{-\hat{\lambda}T} \zeta_{0,T}^{(1-\gamma_Y)\beta_{X,\zeta}} \mathbf{E}_0 \left[ X_{0,T}^{1-\gamma_Y} \zeta_{0,T}^{-(1-\gamma_Y)\beta_{X,\zeta}} \right] \right] \\
&= \zeta_{0,T}^{(1-\gamma_Y)\beta_{X,\zeta}} \mathbf{E}_0 \left[ e^{-(\hat{\lambda} + (1-\gamma_Y)\beta_{X,\zeta} \phi \delta)T} \mathbf{E}_0 \left[ X_{0,T}^{(1-\gamma_Y)(1-\beta_{X,\zeta})} c_{0,T}^{(1-\gamma_Y)\beta_{X,\zeta}} \right] \right] \\
&= \zeta_{0,T}^{(1-\gamma_Y)\beta_{X,\zeta}} \mathbf{E}_0 \left[ e^{-(\hat{\lambda} + (1-\gamma_Y)\beta_{X,\zeta} \phi \delta - \mathbf{b}' \mathbf{M} - \mathbf{b}' \boldsymbol{\Sigma} \mathbf{b} / 2)T} \right] \\
&= \left( \frac{z_0}{z^*} \right)^{\alpha - (1-\gamma_Y)\beta_{X,\zeta}},
\end{aligned} \tag{114}$$

where  $\mathbf{b} = (1-\gamma_Y) (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})$ , and the last line results from  $\zeta_{0,T} = z^*/z_0$  and the definition of  $\alpha$  (equation (37) in the Strategy Hypothesis), and

$$\begin{aligned}
\mathbf{E}_0 \left[ e^{-(\lambda + \phi \delta)T} \xi_{0,T} c_{0,T} \right] &= \mathbf{E}_0 \left[ e^{-(\lambda + \phi \delta)T} \left( e^{-(1-\gamma_Y)\delta T} X_{0,T}^{-\gamma_Y} \right) \left( e^{\phi \delta T} X_{0,T} \zeta_{0,T}^{-1} \right) \right] \\
&= \left( \frac{z_0}{z^*} \right) \mathbf{E}_0 \left[ e^{-\hat{\lambda}T} X_{0,T}^{1-\gamma_Y} \right].
\end{aligned} \tag{115}$$

### Proof of Proposition 4.2

**Lemma B.1.** *If  $x_t$  and  $y_t$  are geometric Brownian processes and  $M_t^y$  denotes the maximum of  $y$  up to time- $t$ , then*

$$\Pi \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-rs} x_{t,s} (M_{t,s}^y)^\Theta ds \mid y_t = M_t^y \right] = \left( \frac{\alpha - \beta}{\alpha - \beta - \Theta} \right) \pi_x \tag{116}$$

where  $\pi_x = \frac{1}{r - \mu_x}$  and

$$\begin{aligned}\alpha &= \sqrt{\left(\frac{\mu_y}{\sigma_y^2} - \frac{1}{2}\right)^2 + \frac{2(r - \mathbf{v}'\mathbf{M} - \mathbf{v}'\boldsymbol{\Sigma}\mathbf{v})}{\sigma_y^2}} - \left(\frac{\mu_y}{\sigma_y^2} - \frac{1}{2}\right) \\ \beta &= \frac{\mathbf{e}_x' \boldsymbol{\Sigma} \mathbf{e}_y}{\mathbf{e}_y' \boldsymbol{\Sigma} \mathbf{e}_y}\end{aligned}$$

where  $\mathbf{e}_x = (1, 0)'$ ,  $\mathbf{e}_y = (0, 1)'$ ,  $\mathbf{v} = \mathbf{e}_x - \beta \mathbf{e}_y$ , and

$$\begin{aligned}\mathbf{M} &\equiv \frac{\mathbf{E}[d\mathbf{z}_t]}{dt} \\ \boldsymbol{\Sigma} &\equiv \frac{\mathbf{E}[d\mathbf{z}_t d\mathbf{z}_t']}{dt}.\end{aligned}$$

where  $\mathbf{z}_t \equiv (\ln x_t, \ln y_t)'$ .

*Proof of lemma:* Given any initial  $x_t$ ,  $y_t$ , and  $M_t^y$

$$\begin{aligned}V(x_t, y_t, M_t^y) &\equiv \mathbf{E}_t \left[ \int_0^\infty e^{-rs} x_{t+s} (M_{t+s}^y)^\ominus ds \right] \\ &= x_t (M_t^y)^\ominus \mathbf{E}_t \left[ \int_0^\infty e^{-rs} x_{t,s} (M_{t,s}^y)^\ominus ds \right].\end{aligned}\quad (117)$$

If  $\tau \equiv \min_{s>0} \{y_{t+s} = M_t^y\}$ , then  $M_{t+s}^y = M_t^y$  on the interval  $(t, t + \tau)$  so

$$\begin{aligned}\mathbf{E}_t \left[ \int_0^\infty e^{-rs} x_{t,s} (M_{t,s}^y)^\ominus ds \right] &= \mathbf{E}_t \left[ \int_0^\infty e^{-rs} x_{t,s} ds \right] + \mathbf{E}_t \left[ \int_\tau^\infty e^{-rs} (x_{t,s} (M_{t,s}^y)^\ominus - x_{t,s}) ds \right] \\ &= \pi + \mathbf{E}_t [e^{-r\tau} x_{t,\tau}] (\Pi - \pi).\end{aligned}\quad (118)$$

Now  $x_{t,s}/y_{t,s}^\beta$  is independent of  $y_{t,s}$ , and thus independent of  $\tau$ , so

$$\begin{aligned}\mathbf{E}_t [e^{-r\tau} x_{t,\tau}] &= \mathbf{E}_t \left[ e^{-r\tau} y_{t,\tau}^\beta \mathbf{E} \left[ x_{t,\tau} y_{t,\tau}^{-\beta} \right] \right] \\ &= y_{t,\tau}^\beta \mathbf{E}_t \left[ e^{-(r - \mathbf{v}'\mathbf{M} - \mathbf{v}'\boldsymbol{\Sigma}\mathbf{v})\tau} \right] \\ &= y_{t,\tau}^{\beta - \alpha}.\end{aligned}\quad (119)$$

So, using  $y_{t,\tau} = M_t^y / y_t$ ,

$$V(x_t, y_t, M_t^y) = x_t (M_t^y)^\ominus \left( \pi + \left( \frac{M_t^y}{y_t} \right)^{\beta - \alpha} (\Pi - \pi) \right).\quad (120)$$

Smooth pasting requires  $\frac{d}{dy}V(x, y, M^y)|_{y=M^y} = \frac{d}{dM^y}V(x, M^y, M^y)$ , and

$$\begin{aligned} \frac{d}{dy}V(x, y, M^y)|_{y=M^y} &= (M^y)^\Theta \left( \frac{dx}{dy}\pi + \left( \frac{dx}{dy} + \frac{x(\alpha - \beta)}{y} \right) \left( \frac{M^y}{y} \right)^{\beta - \alpha} (\Pi - \pi) \right) \Big|_{y=M^y} \\ &= x (M^y)^{\Theta - 1} (\alpha \Pi - (\alpha - \beta) \pi) \end{aligned} \quad (121)$$

where we've used  $\frac{dx}{dy} = \beta x/y$ , while

$$\begin{aligned} \frac{d}{dM^y}V(x, M^y, M^y) &= \left( \frac{dx}{dM^y} (M^y)^\Theta + x \frac{d(M^y)^\Theta}{dM^y} \right) \Pi \\ &= x (M^y)^{\Theta - 1} (\beta + \Theta) \Pi. \end{aligned} \quad (122)$$

Equating the right hand sides of equations (121) and (122) and solving for  $\Pi$  yields the lemma. ■

*Proof of the proposition:*  $\mathbf{E}_t \left[ e^{-\hat{\lambda}\tau} X_{t,\tau}^{1-\gamma_Y} \right] = \zeta_{t,\tau}^{(1-\gamma_Y)\beta_{X,\xi} - \alpha}$  follows directly from equation (114). The perpetuity factor follows from Lemma B.1,

$$\begin{aligned} \Pi^{II} &\equiv \mathbf{E}_t \left[ \int_0^\infty e^{-\hat{\lambda}s} X_{t,s}^{1-\gamma_Y} M_{t,s}^{\Omega + \Psi - 1} ds \mid \zeta_t = M_t \right] \\ &= \left( \frac{\alpha - (1 - \gamma_Y) \beta_{X,\xi}}{\alpha - (1 - \gamma_Y) \beta_{X,\xi} - (\Omega + \Psi - 1)} \right) \mathbf{E}_0 \left[ \int_0^\infty e^{-\hat{\lambda}t} X_{0,t}^{1-\gamma_Y} dt \right] \\ &= \left( \frac{\Psi\eta + \Omega - 1}{\Psi\eta - \Psi} \right) \pi \end{aligned} \quad (123)$$

where the third line uses  $\Psi\eta = \alpha + 1 - (1 - \gamma_Y) \beta_{X,\xi} - \Omega$ , and the definitions of  $\alpha$  and  $\pi$  given in proposition 4.1. Finally, again using  $\Psi\eta = \alpha + 1 - (1 - \gamma_Y) \beta_{X,\xi} - \Omega$ , we have

$$\begin{aligned} \mathbf{E}_t \left[ e^{-(\lambda + \phi\delta)\tau} \xi_{t,\tau} c_{t,\tau} \right] &= \mathbf{E}_t \left[ e^{-(\lambda + \phi\delta)\tau} \left( e^{-(1-\gamma_Y)\delta\tau} X_{t,\tau}^{-\gamma_Y} M_{t,\tau}^\Omega \right) \left( \frac{e^{\phi\delta\tau} X_{t,\tau}}{\zeta_{t,\tau}} \right) \right] \\ &= \zeta_{t,\tau}^{\Omega - 1} \mathbf{E}_t \left[ e^{-\hat{\lambda}\tau} X_{t,\tau}^{1-\gamma_Y} \right] \\ &= \left( \frac{\zeta_t}{\zeta^*} \right)^{\Psi\eta}. \end{aligned} \quad (124)$$

### Proof of Proposition 4.3

**Lemma B.2.** *If  $f(1) > 0$ ,  $f'(1) > 0$ , and  $f''(x) \geq 0$  for all  $x > 1$ , then for any  $a > 1$*

$$\frac{d^2}{dx^2} f(x)^a > 0 \quad \text{for all } x > 1 \quad (125)$$

$$\lim_{x \rightarrow \infty} \frac{x}{f(x)^a} = 0. \quad (126)$$

*Proof of lemma:* For the convexity of  $f(x)^a$ ,

$$\frac{d^2}{dx^2} f(x)^a = a(a-1)f(x)^{a-2} f'(x)^2 + a f(x)^{a-1} f''(x), \quad (127)$$

and for  $x > 1$  the first term on the right is strictly positive, while the second term is non-negative.

The fact that  $o(f(x)^a) > 1$  follows immediately from the fact that  $f$  is positive and convex, which implies that  $f(x) \geq f'(1)(x-1)$ .

*Proof of the proposition:* Let  $f(x) = K(x)^{1-1/\eta}$ , and let

$$x^* = \arg \max_{x>1} \left\{ h(x) \equiv \left( \frac{x-1}{f(x)} \right)^\eta \right\}, \quad (128)$$

which exist because  $h(x)$  is continuous and positive on  $x > 1$  and  $h(1) = 0 = \lim_{x \rightarrow \infty} h(x)$  (by lemma B.2), and is unique because  $x^*$  must satisfy

$$f(x) - (x-1)f'(x) = 0, \quad (129)$$

which has at most one solution because the left hand side of which is strictly positive at  $x = 1$ , and decreasing for all  $x > 1$  because  $f(x)$  is strictly convex on  $x > 1$ , by lemma B.2, using the weak convexity of  $K(x)^{1/\phi}$  and  $(\eta-1)\phi/\eta > 1$ .

Now

$$\kappa = \arg \max_{x>1} \left\{ g(x) \equiv \left( \frac{1}{1-x^{\eta-\phi-\eta\phi}} \right) \left( \frac{x-1}{f(x)} \right)^\eta \right\} \quad (130)$$

must satisfy  $\frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)} = 0$  because  $g(x) > 0$  for  $x > 1$ , and

$$\frac{d}{dx} \ln g(x) = \eta \left( \frac{1}{x-1} - \frac{f'(x)}{f(x)} \right) - \frac{\eta\phi - \eta - \phi}{x^{(\eta-1)(\phi-1)} - x}, \quad (131)$$

has a unique solution greater than one, because it is strictly positive (unbounded) in the limit as  $x$  goes to one from above, strictly negative for all  $x > x^*$  because  $\frac{d}{dx} \ln g(x) < \frac{d}{dx} \ln h(x) < 0$  for  $x > x^*$ , and strictly decreasing for all  $x \in (1, x^*)$ , because the second term on the right hand side of the previous equation is decreasing for all  $x > 1$ , and the derivative of the first term is

$$\frac{\eta \frac{d}{dx} (f(x) - (x-1)f'(x))}{(x-1)f(x)} - \frac{\eta (f(x) - (x-1)f'(x)) \frac{d}{dx} ((x-1)f(x))}{(x-1)^2 f(x)^2} \quad (132)$$

which is strictly negative on the interval  $(1, x^*)$ , because the first term is negative for  $x > 1$  and the second term of is negative for  $x \in (1, x^*)$ .

#### **Proof of Proposition 4.4**

*Proof of the proposition:* We already calculated the annuity factor for housing capital, equation (123). Cal-



culating the annuity factor for non-housing capital follows the same methodology,

$$\begin{aligned}
\Pi^I &\equiv \mathbf{E}_0 \left[ \int_0^\infty e^{-\lambda t} \xi_{0,t} X_{0,t} dt \right] \\
&= \mathbf{E}_0 \left[ \int_0^\infty e^{-\hat{\lambda} t} X_{0,t}^{1-\gamma_Y} M_{0,t}^\Omega dt \right] \\
&= \left( \frac{\alpha - (1 - \gamma_Y) \beta_{X,\xi}}{\alpha - (1 - \gamma_Y) \beta_{X,\xi} - \Omega} \right) \pi \\
&= \left( \frac{\Psi \eta + \Omega - 1}{\Psi \eta - 1} \right) \pi
\end{aligned} \tag{133}$$

Then

$$\frac{\eta \Pi^I}{(\eta - 1) \Pi^{II}} = \frac{\eta(\Psi \eta - \Psi)}{(\eta - 1)(\Psi \eta - 1)} = \frac{1}{1 - \frac{1}{\Psi \eta}}. \tag{134}$$

#### Proof of Proposition 4.5

*Proof of the proposition:* Using the results of Proposition 4.1,

$$\frac{V(q, P, c, z)}{qP} = \pi + \left( \frac{z_0}{z^*} \right)^{\Psi \eta + \Omega - 1} \left( \frac{V\left(q, \left(\frac{z^*}{z}\right) P, c, z^*\right)}{q \left(\frac{z^*}{z}\right) P} - \pi \right). \tag{135}$$

Then proposition 4.2 and the definition of  $a_q$  gives

$$\frac{V\left(q, \left(\frac{z^*}{z}\right) P, c, z^*\right)}{q \left(\frac{z^*}{z}\right) P} = \Pi^{II} + q^{-1+\eta+\phi-\eta\phi} \left( \frac{\kappa-1}{\eta(1-\kappa\eta+\phi-\eta\phi)} \right) \Pi^{II} \left( \frac{p}{p^*} \right)^{\eta-1}. \tag{136}$$

Substituting the previous equation into the preceding equation, together with the fact that at the development boundary  $p = \underline{p}^{\phi-1} p^*$ , then yields the proposition.

#### Proof of Proposition 6.1

**Lemma B.3.** *Suppose  $x_t$  is a geometric Brownian process with volatility  $\sigma$  and expected growth rate  $\mu > \sigma^2/2$ , and let  $M_t^x$  denote the maximum of  $x$  up to time- $t$ . Then for  $T > 0$  and arbitrary  $\alpha$  and  $\beta$*

$$\mathbf{E}_t \left[ x_{t,T}^\alpha (M_{t,T}^x)^\beta \right] = F_x \left( \frac{M_t^x}{x_t}, \alpha, \beta, T \right) \mathbf{E}_t \left[ x_{t,T}^\alpha \right] + G_x \left( \frac{M_t^x}{x_t}, \alpha, \beta, T \right) \mathbf{E}_t \left[ x_{t,T}^{\alpha+\beta} \right] \tag{137}$$

where

$$F_x(X, \alpha, \beta, T) = N \left( \frac{\ln X}{\sigma \sqrt{T}} - c \sqrt{T} \right) - \theta X^{2c/\sigma} N \left( \frac{-\ln X}{\sigma \sqrt{T}} - c \sqrt{T} \right) \tag{138}$$

$$G_x(X, \alpha, \beta, T) = (1 + \theta) X^{-\beta} N \left( \frac{-\ln X}{\sigma \sqrt{T}} + (c + \beta \sigma) \sqrt{T} \right) \tag{139}$$

with  $c \equiv \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right) + \alpha\sigma$  and  $\theta \equiv \frac{\sigma\beta}{2c + \sigma\beta}$ .

Before proving the lemma, we would like to note some comforting properties of this explicit characterization. First, if  $\beta = 0$  then the right hand side of equation (137) reduces to  $\mathbf{E}_t \left[ x_{t,T}^\alpha \right]$ , as it obviously should. The right hand side of (137) also reduces to  $\mathbf{E}_t \left[ x_{t,T}^\alpha \right]$  in the limit as  $M^x/x$  gets large. Finally, if  $\alpha = -\beta = 1$  then as  $T$  gets large we get the well known long-run mean of a geometric Brownian process reflected from above,

$$\lim_{T \rightarrow \infty} x_t \mathbf{E}_t \left[ x_{t,T} / M_{t,T}^x \right] = \left( \frac{\mu - \frac{\sigma^2}{2}}{\mu} \right) M_t^x. \quad (140)$$

*Proof of lemma:* The underlying process  $x_{t,T}$  is distributed  $\exp \left[ \sigma \left( \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) T + \sqrt{T} \chi \right) \right]$  where  $\chi$  denotes a standard normal random variable. Using this, in conjunction with the Markovian nature of Brownian motion and the joint density for the value and the maximum of a standard Brownian up to time  $T$ , and a change of measure, we have that the expectation on the left hand side of the previous equation is given by

$$\int_{m=0}^{\infty} \int_{b=-\infty}^m e^{\sigma b \alpha} \left( \max \left( 1, \frac{x_t e^{\sigma m}}{M_t^x} \right) \right)^\beta e^{\left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) b - \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2 \frac{T}{2}} v(b, m) db dm \quad (141)$$

where  $v(b, m) = \sqrt{\frac{2}{\pi}} \left( \frac{2m-b}{T\sqrt{T}} \right) e^{-\frac{(2m-b)^2}{2T}}$  is the joint density for the value and the maximum of a standard Brownian process at  $T$ . Rearranging the previous equation using the definition of  $c$  yields

$$e^{-\left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2 \frac{T}{2}} \int_{m=0}^{\infty} \left( \max \left( 1, \frac{x_t e^{\sigma m}}{M_t^x} \right) \right)^\beta \int_{b=-\infty}^m e^{cb} v(b, m) db dm. \quad (142)$$

The interior integral, after completing the square in the exponent and simplifying, yields

$$e^{2cm + c^2 T/2} \int_{b=-\infty}^m \sqrt{\frac{2}{\pi}} \left( \frac{2m-b}{T\sqrt{T}} \right) \exp \left( \frac{-(b - (2m + cT))^2}{2T} \right) db. \quad (143)$$

Using  $\frac{2m-b}{T\sqrt{T}} = \frac{-1}{\sqrt{T}} \left( \frac{b - (2m + cT)}{T} \right) - \frac{c}{\sqrt{T}}$  the integral becomes

$$\begin{aligned} & -\sqrt{\frac{2}{\pi T}} \int_{b=-\infty}^m \left( \frac{b - (2m + cT)}{T} \right) \exp \left( \frac{-(b - (2m + cT))^2}{2T} \right) db \\ & -2c \int_{b=-\infty}^m \frac{1}{\sqrt{2\pi T}} \exp \left( \frac{-(b - (2m + cT))^2}{2T} \right) db \\ & = \sqrt{\frac{2}{\pi T}} \exp \left( \frac{-(m + cT)^2}{2T} \right) - 2c N \left( \frac{-m - cT}{\sqrt{T}} \right). \end{aligned} \quad (144)$$

Substituting the previous two equations into equation (142), we have that  $\mathbf{E}_t \left[ (x_{t,T})^\alpha \left( M_{t,T}^x \right)^\beta \right]$  is equal to

$$e^{\alpha \left( \mu - \frac{(1-\alpha)\sigma^2}{2} \right) T} \int_{m=0}^{\infty} \left( \max \left( 1, \frac{x_t e^{\sigma m}}{M_t^x} \right) \right)^\beta e^{2cm} \left( \sqrt{\frac{2}{\pi T}} \exp \left( \frac{-(m + cT)^2}{2T} \right) - 2c N \left( \frac{-m - cT}{\sqrt{T}} \right) \right) dm \quad (145)$$

We can evaluate the integral in the previous equation by considering the regions  $m < \frac{1}{\sigma} \ln(M_t^x/x_t)$  and  $m > \frac{1}{\sigma} \ln(M_t^x/x_t)$  separately. Over the region  $m < \frac{1}{\sigma} \ln(M_t^x/x_t)$ , integrating the first term by completing the square in the exponent and the second term by parts, we have

$$\begin{aligned}
& \int_0^{\frac{1}{\sigma} \ln(M_t^x/x_t)} e^{2cm} \left( \sqrt{\frac{2}{\pi T}} \exp\left(\frac{-(m+cT)^2}{2T}\right) - 2cN\left(\frac{-m-cT}{\sqrt{T}}\right) \right) dm \\
&= \int_0^{\frac{1}{\sigma} \ln(M_t^x/x_t)} \frac{\exp\left(\frac{-(m+cT)^2}{2T}\right)}{\sqrt{2\pi T}} dm - \left[ e^{2cm} N\left(\frac{-m-cT}{\sqrt{T}}\right) \right]_{m=0}^{\frac{1}{\sigma} \ln(M_t^x/x_t)} \\
&= N\left(\frac{\ln(M_t^x/x_t)}{\sigma\sqrt{T}} - c\sqrt{T}\right) - N\left(\frac{-\ln(M_t^x/x_t)}{\sigma\sqrt{T}} - c\sqrt{T}\right) \left(\frac{M_t^x}{x_t}\right)^{2c/\sigma}.
\end{aligned} \tag{146}$$

Over the region  $m > \frac{1}{\sigma} \ln(M_t^x/x_t)$ , again integrating the first term by completing the square in the exponent and the second term by parts, we have

$$\begin{aligned}
& \left(\frac{x_t}{M_t^x}\right)^\beta \int_{\frac{1}{\sigma} \ln(M_t^x/x_t)}^\infty e^{(2c+\sigma\beta)m} \left( \sqrt{\frac{2}{\pi T}} \exp\left(\frac{-(m+cT)^2}{2T}\right) - 2cN\left(\frac{-m-cT}{\sqrt{T}}\right) \right) dm \\
&= \left(\frac{x_t}{M_t^x}\right)^\beta \left( \left(2 - \frac{2c}{2c+\sigma\beta}\right) \int_{\frac{1}{\sigma} \ln(M_t^x/x_t)}^\infty e^{(2c+\sigma\beta)m} \frac{\exp\left(\frac{-(m-(c+\sigma\beta)T)^2}{2T}\right)}{\sqrt{2\pi T}} dm \right. \\
&\quad \left. - \left(\frac{2c}{2c+\sigma\beta}\right) \left[ e^{(2c+\sigma\beta)m} N\left(\frac{-m-cT}{\sqrt{T}}\right) \right]_{\frac{1}{\sigma} \ln(M_t^x/x_t)}^\infty \right) \\
&= (1+\theta)N\left(\frac{-\ln(M_t^x/x_t)}{\sigma\sqrt{T}} + (c+\sigma\beta)\sqrt{T}\right) e^{\sigma\beta(2c+\sigma\beta)T/2} \left(\frac{x_t}{M_t^x}\right)^\beta \\
&\quad + (1-\theta)N\left(\frac{-\ln(M_t^x/x_t)}{\sigma\sqrt{T}} - c\sqrt{T}\right) \left(\frac{M_t^x}{x_t}\right)^{\frac{2c}{\sigma}}
\end{aligned} \tag{147}$$

where  $\theta = \frac{\sigma\beta}{2c+\sigma\beta}$ , and we have used the fact that  $N(-x) = 1 - N(x)$ .

Substituting the previous two equations into equation (145) yields

$$\begin{aligned}
& e^{\alpha(\mu+(\alpha-1)\frac{\alpha^2}{2})T} \left( N\left(\frac{\ln\frac{M_t^x}{x_t}}{\sigma\sqrt{T}} - c\sqrt{T}\right) - \theta N\left(\frac{-\ln\frac{M_t^x}{x_t}}{\sigma\sqrt{T}} - c\sqrt{T}\right) \left(\frac{M_t^x}{x_t}\right)^{2c/\sigma} \right. \\
& \left. + (1+\theta)N\left(\frac{-\ln\frac{M_t^x}{x_t}}{\sigma\sqrt{T}} + (c+\beta\sigma)\sqrt{T}\right) e^{\beta(\mu+(\beta-1)\frac{\alpha^2}{2})T + \alpha\beta\sigma^2 T} \left(\frac{x_t}{M_t^x}\right)^\beta \right),
\end{aligned} \tag{148}$$

which proves the lemma.

**Lemma B.4.** For  $T > 0$  and arbitrary  $a$  and  $b$

$$\mathbf{E}_t \left[ X_{t,T}^a M_{t,T}^b \right] = \mathbf{E}_t \left[ X_{t,T}^a \right] \exp\left(-H_\zeta\left(\frac{M_t}{\xi_t}, a\beta_{x,\zeta}, b, T\right) T\right) \tag{149}$$

where  $M_t$  denotes the maximum of  $\zeta$  up to time- $t$ , and

$$H(X, a, b, T) \equiv \frac{1}{T} \ln \left( F_\zeta(X, a, b, T) + e^{\Lambda(a,b)T} G_\zeta(X, a, b, T) \right) \quad (150)$$

$$\begin{aligned} \Lambda(a, b) &\equiv \frac{1}{T} \left( \ln \mathbf{E}_t \left[ \zeta_{t,T}^{a+b} \right] - \ln \mathbf{E}_t \left[ \zeta_{t,T}^a \right] \right) \\ &= b \left( \phi \delta + \mathbf{1}' \mathbf{M} + (b + 2a) \frac{\mathbf{1}' \Sigma \mathbf{1}}{2} \right). \end{aligned} \quad (151)$$

*Proof of lemma:*

$$\mathbf{E}_t \left[ X_{t,T}^a M_{t,T}^b \right] = \mathbf{E}_t \left[ X_{t,T}^a \zeta_{t,T}^{-a\beta_{X,\zeta}} \right] \mathbf{E}_t \left[ \zeta_{t,T}^{a\beta_{X,\zeta}} M_{t,T}^b \right],$$

while by lemma B.3

$$\mathbf{E}_t \left[ \zeta_{t,T}^{a\beta_{X,\zeta}} M_{t,T}^b \right] = \mathbf{E}_t \left[ \zeta_{t,T}^{a\beta_{X,\zeta}} \right] \exp \left( H_\zeta \left( \frac{M_t}{\zeta_t}, a\beta_{X,\zeta}, b, T \right) T \right) \quad (152)$$

and

$$\begin{aligned} &\frac{1}{T} \left( \ln \mathbf{E}_t \left[ X_{t,T}^a \zeta_{t,T}^{-a\beta_{X,\zeta}} \right] + \ln \mathbf{E}_t \left[ \zeta_{t,T}^{a\beta_{X,\zeta}} \right] \right) \\ &= \frac{1}{T} \left( \ln \mathbf{E}_t \left[ X_{t,T}^{a(1-\beta_{X,\zeta})} c_{t,T}^{a\beta_{X,\zeta}} \right] + \ln \mathbf{E}_t \left[ (X_{t,T}/c_{t,T})^{a\beta_{X,\zeta}} \right] \right) \\ &= a (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \mathbf{M} + \frac{a^2}{2} (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \Sigma (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) \\ &\quad + a ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \mathbf{M} + \frac{a^2}{2} ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \Sigma ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) \\ &= a \mathbf{e}'_X \mathbf{M} + a^2 \frac{\mathbf{e}'_X \Sigma \mathbf{e}_X}{2}, \end{aligned} \quad (153)$$

which is the growth rate of  $\mathbf{E}_t \left[ X_{t,T}^a \right]$ , where the last line follows from

$$\begin{aligned} (\mathbf{e}'_X \boldsymbol{\beta})' \Sigma (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) &= (\mathbf{e}_X - (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \Sigma ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) \\ &= \mathbf{e}'_X \Sigma ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) - ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1})' \Sigma ((\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{1}) \\ &= (\mathbf{e}'_X \boldsymbol{\beta}) \mathbf{e}'_X \Sigma \mathbf{1} - (\mathbf{e}'_X \boldsymbol{\beta})^2 \mathbf{1}' \Sigma \mathbf{1}, \end{aligned} \quad (154)$$

which is zero because  $\boldsymbol{\beta} = \frac{\Sigma \mathbf{1}}{\mathbf{1}' \Sigma \mathbf{1}}$ .

*Proof of the proposition:*  $g_t(T) = \ln \mathbf{E}_t [C_{t,T}] / T$  and  $C_{t,T} = e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b$ , so

$$r_t^f(T) = \frac{1}{T} \ln \mathbf{E}_t \left[ X_{t,T}^a M_{t,T}^b \right] - (1-a)\delta, \quad (155)$$

and the proposition follows directly from  $\frac{1}{T} \ln \mathbf{E}_t \left[ X_{t,T}^a \right] = a \left( \mu_X + (a-1) \frac{\sigma_X^2}{2} \right)$  using Lemma B.4.

For corollary 6.2,

$$\lim_{T \rightarrow \infty} g_t(T) = a \left( \mu_X + (a-1) \frac{\sigma_X^2}{2} \right) - (1-a)\delta + \Lambda(a\beta_{X,\xi}, b) \quad (156)$$

where  $\Lambda(a\beta_{X,\xi}, b) = b(\phi\delta + \mathbf{1}'\mathbf{M} + (b + 2a\beta_{X,\xi})\mathbf{1}'\Sigma\mathbf{1}/2)$ , so

$$\mu_\infty^C = \mathbf{v}'\mathbf{M} + \frac{\mathbf{v}'\Sigma\mathbf{v}}{2} - (1-a-b\phi)\delta \quad (157)$$

where  $\mathbf{v} = a\mathbf{e} + b\mathbf{1}$ . Using  $b\phi/(1-a) = \phi/(\phi-1)\eta$  and the definition of  $\Psi$  then proves the corollary.

### Proof of Proposition 7.1

*Proof of the proposition:*  $B_t^f(T) = \mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C \right]$  and  $\xi_{t,T}^C = e^{(1-a)\gamma_C \delta T} X_{t,T}^{-a\gamma_C} M_{t,T}^{-b\gamma_C}$ , so

$$r_t^f(T) = \lambda - (1-a)\gamma_C \delta - \frac{1}{T} \ln \mathbf{E}_t \left[ X_{t,T}^{-a\gamma_C} M_{t,T}^{-b\gamma_C} \right], \quad (158)$$

and the proposition follows directly from  $\frac{1}{T} \ln \mathbf{E}_t \left[ X_{t,T}^{-a\gamma_C} \right] = -a\gamma_C \left( \mu_X - (1+a\gamma_C) \frac{\sigma_X^2}{2} \right)$  using Lemma B.4 and .

For corollary 7.2,

$$\lim_{T \rightarrow \infty} r_t^f(T) = \lambda - (1-a)\gamma_C \delta + a\gamma_C \left( \mu_X - (1+a\gamma_C) \frac{\sigma_X^2}{2} \right) - \Lambda(-a\gamma_C \beta_{X,\xi}, -b\gamma_C) \quad (159)$$

where  $\Lambda(-a\gamma_C \beta_{X,\xi}, -b\gamma_C) = -b\gamma_C \left( \phi\delta + \mathbf{1}'\mathbf{M} - (b + 2a\beta_{X,\xi})\gamma_C \frac{\mathbf{1}'\Sigma\mathbf{1}}{2} \right)$ , so

$$\lim_{T \rightarrow \infty} r_{t,T}^f = r_{L(a\gamma_C)}^f - (1-a-b\phi)\gamma_C \delta + b\gamma_C \left( \mathbf{1}'\mathbf{M} - (b + 2a\beta_{X,\xi})\gamma_C \frac{\mathbf{1}'\Sigma\mathbf{1}}{2} \right). \quad (160)$$

### Proof of Proposition 7.2

*Proof of the proposition:* The expected rate of return over the life of the consumption bond is given by  $r_t^c(T) = \frac{1}{T} \left( \ln \mathbf{E}_t [C_{t,T}] - \ln \mathbf{E}_t \left[ e^{-\lambda T} C_{t,T}^{1-\gamma_C} \right] \right)$ , and using Lemma B.4 and  $C_{t,T} = e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b$

$$\begin{aligned} \mathbf{E}_t [C_{t,T}] &= \mathbf{E}_t \left[ e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b \right] \\ &= e^{-(1-a)\delta T} \mathbf{E}_t \left[ X_{t,T}^a \right] \exp \left( H \left( \frac{z^*}{z_t}, a\beta_{X,\xi}, b, T \right) T \right). \end{aligned} \quad (161)$$

while

$$\begin{aligned} \mathbf{E}_t \left[ e^{-\lambda T} C_{t,T}^{1-\gamma_C} \right] &= \mathbf{E}_t \left[ e^{-(\lambda+(1-a)(1-\gamma_C)\delta)T} \left( X_{t,T}^a M_{t,T}^b \right)^{1-\gamma_C} \right] \\ &= e^{-\hat{\lambda}T} \mathbf{E}_t \left[ X_{t,T}^{a(1-\gamma_C)} \right] \exp \left( H \left( \frac{z^*}{z_t}, a(1-\gamma_C)\beta_{X,\xi}, b(1-\gamma_C), T \right) T \right). \end{aligned} \quad (162)$$

Taking  $\frac{1}{T} \left( \ln \mathbf{E}_t \left[ X_{t,T}^a \right] - \ln \mathbf{E}_t \left[ X_{t,T}^{a(1-\gamma_C)} \right] \right) = a\gamma_C \left( \mu_X - (1 + a\gamma_C) \frac{\sigma_X^2}{2} \right) + \gamma_C a^2 \sigma_X^2$  together with the previous two equations then yields the proposition.

For corollary 7.4, note that  $r_\infty^c = r_\infty^f + \lim_{T \rightarrow \infty} r_t^e(T)$ ,  $\gamma_C \sigma_C^2 = \gamma_C a^2 \sigma_X^2$ , and for arbitrary  $a_1, a_2, b_1$  and  $b_2$

$$\Lambda(a_1 + a_2, b_1 + b_2) - \Lambda(a_1, b_2) - \Lambda(a_2, b_1) = (a_1 b_1 + a_2 b_2 + b_1 b_2) \mathbf{1}' \Sigma \mathbf{1}, \quad (163)$$

so

$$\begin{aligned} \Lambda(a(1-\gamma_C)\beta_{X,\xi}, b(1-\gamma_C)) - \Lambda(a\beta_{X,\xi}, b) - \Lambda(-a\gamma_C\beta_{X,\xi}, -b\gamma_C) \\ = -\gamma_C b(b + 2a\beta_{X,\xi}) \mathbf{1}' \Sigma \mathbf{1}, \end{aligned} \quad (164)$$

and  $\sigma_X^2 + b(2a\beta_{X,\xi} + b) \mathbf{1}' \Sigma \mathbf{1} = (a\mathbf{e}_X + b\mathbf{1})' \Sigma (a\mathbf{e}_X + b\mathbf{1})$ .

### Proof of Proposition 7.3

*Proof of the proposition:*  $B_t^i(T) = \mathbf{E}_t \left[ e^{-\lambda T} \xi_{t,T}^C P_{t,T}^i \right]$  where  $P_{t,T}^Q = \frac{C_{t,T}}{Q_{t,T}} = P_{t,T}^a$  and  $P_{t,T}^Y = P_{t,T} \cdot P_{t,T}^Q = P_{t,T}^{a-1}$ , so

$$\begin{aligned} B_t^Y(T) &= \mathbf{E}_t \left[ e^{-\lambda T} \left( e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b \right)^{-\gamma_C} \left( e^{\delta T} X_{t,T} M_{t,T}^{\Psi-1} \right)^{a-1} \right] \\ &= e^{-(\lambda+(1-a)(1-\gamma_C)\delta)T} \mathbf{E}_t \left[ X_{t,T}^{a(1-\gamma_C)-1} M_{t,T}^{(1-a)(1-\Psi)-b\gamma_C} \right] \\ &= e^{-(\lambda+(1-\gamma_Q)\delta)T} \mathbf{E}_t \left[ X_{t,T}^{-\gamma_Y} M_{t,T}^\Omega \right] \end{aligned} \quad (165)$$

and

$$\begin{aligned} B_t^Q(T) &= \mathbf{E}_t \left[ e^{-\lambda T} \left( e^{-(1-a)\delta T} X_{t,T}^a M_{t,T}^b \right)^{-\gamma_C} \left( e^{\delta T} X_{t,T} M_{t,T}^{\Psi-1} \right)^a \right] \\ &= e^{-(\lambda - ((1-a)\gamma_C + a)\delta)T} \mathbf{E}_t \left[ X_{t,T}^{a(1-\gamma_C)} M_{t,T}^{a(\Psi-1)-b\gamma_C} \right] \\ &= e^{-(\lambda - \gamma_Q \delta)T} \mathbf{E}_t \left[ X_{t,T}^{1-\gamma_Y} M_{t,T}^{\Omega+\Psi-1} \right]. \end{aligned} \quad (166)$$

The proposition then follows from  $r_t^i(T) = -(\ln B_t^i(T))/T$  and Lemma B.4.

### Proof of Proposition 8.1

*Proof of the proposition:* To see that  $1 - w\left(\frac{z}{z^*}, 0, \Omega\right) = \pi / \Pi^I\left(\frac{z}{z^*}\right)$ , note that

$$\frac{\Pi^I\left(\frac{z}{z^*}\right)}{\pi} = 1 + \left(\frac{z}{z^*}\right)^{\Psi\eta + \Omega - 1} \left(\frac{\Pi^I}{\pi} - 1\right). \quad (167)$$

Taking the reciprocal and substituting  $\Pi^I/\pi = 1 + \Omega/(\Psi\eta - 1)$ , given in equation (64), then yields  $1 -$

$w\left(\frac{z_t}{z^*}, 0, \Omega\right)$ . Also,

$$\begin{aligned}\pi^{-1} &= \lambda - (1 - \gamma_C) \left( a \left( \mu_X + (a - 1) \frac{\sigma_X^2}{2} \right) - (1 - a)\delta - \gamma_C a^2 \frac{\sigma_X^2}{2} \right) \\ &= \mathbf{E} \left[ \frac{dC^{1-\gamma_C}}{C^{1-\gamma_C}} \right] / dt,\end{aligned}\tag{168}$$

so using the definition of a consumption bond we have that  $\pi^{-1} + \mu_C = r_0^c$ . Finally, using  $C_{t,s} = e^{(a-1)\delta s} X_{t,s}^a$  and  $\zeta_{t,s} = e^{\phi\delta s} X_{t,s}/c_{t,s}$ , and the definitions of  $\mathbf{M}$  and  $\mathbf{\Sigma}$ , yields  $R$ .

### Proof of Proposition 8.2

*Proof of the proposition:* To see that  $w\left(\frac{z_t}{z^*}, \left(\frac{q}{q}\right)^{(\eta-1)(\phi-1)} \Theta, \Omega - 1 + \Psi\right) = \pi / \Pi^{II}\left(\frac{q}{q}, \frac{z}{z^*}\right)$ , note that

$$\frac{\Pi^{II}\left(\frac{q}{q}, \frac{z}{z^*}\right)}{\pi} = 1 + \left(\frac{z}{z^*}\right)^{\Psi\eta + \Omega - 1} \left( \left(\frac{\Pi^{II}}{\pi} - 1\right) + \Theta \left(\frac{q}{q}\right)^{(\eta-1)(\phi-1)} \frac{\Pi^{II}}{\pi} \right)\tag{169}$$

Taking the reciprocal and substituting  $\Pi^{II}/\pi = 1 + (\Omega - 1 + \Psi) / (\Psi\eta - \Psi)$ , given in equation (49), then yields  $1 - w\left(\frac{z_t}{z^*}, \left(\frac{q}{q}\right)^{(\eta-1)(\phi-1)} \Theta, \Omega - 1 + \Psi\right)$ .

## C Money

We were interested in a numeraire that unambiguously related consumption to consumers' quality of life (*i.e.*, instantaneous utility), and this led us to employ aggregated consumption. This choice is consistent with including money explicitly in the economy, and using “real” dollars, defined as nominal dollars adjusted for inflation using the chained Fisher index, as numeraire. This choice of price deflator is compelling on both practical and theoretical grounds. It is standard in practice, used, for example, by the U.S. Department of Commerce's Bureau of Economic Analysis when compiling the National Income and Product Accounts. It is also “ideal” in our economy, as the unit price of the aggregate consumption bundle  $C_t$  is constant in “real” (*i.e.*, deflated) dollars, and consumption is thus unambiguously related to consumers' quality of life.

### C.1 The Price Deflator

Assume exchange is mediated, due to a cash-in-advance constraint on consumption goods, by currency (fiat money). That is, firms sell their output in exchange for “dollars,” which exist in fixed positive net supply  $m$  and are initially endowed to households, and pay dollar profits as dividends.<sup>15</sup> Calculating the continuously chained Fisher index explicitly yields the following proposition.

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<sup>15</sup> We will make parameter restrictions to ensure that the cash-in-advance constraint binds at all times, though these can be relaxed by introducing sufficient growth in the money supply.

**Proposition C.1.** *The continuously chained Fisher index is ideal; the change in the price level from  $t$  to  $t + T$ , as measured by the chained Fisher index, is the inverse of the change in aggregate consumption,*

$$i_{t,T}^F = C_{t,T}^{-1}. \quad (170)$$

Proposition C.1 says that consumption measured in nominal dollars at prices deflated using the continuously chained Fisher index is directly proportional to aggregate consumption,  $C_t$ . A real dollar buys a basket, optimal at the prevailing prices, which delivers a constant quantity of the appropriately aggregated consumption bundle. That is, at deflated prices a dollar affords an agent with the same preferences as the representative consumer the same quality of life at any time.

### Proof of Proposition C.1

*Proof of the proposition:* The Fisher index attempts to accommodate substitution in consumer spending while holding living standards constant by taking the geometric mean of Laspeyres and Paasche indices, which individually overstate and understate inflation, respectively.

The Laspeyres index measures today's cost of yesterday's consumption basket relative to yesterday's cost of yesterday's basket. That is, letting  $i^L(t, t_0)$  denote the time- $t$  Laspeyres index with base year  $t_0$ , which measures the time- $t$  price of the time- $t_0$  consumption basket relative to the price of the basket at time- $t_0$ ,

$$i^L(t, t_0) = \frac{Y_{t_0} P_t^Y + Q_{t_0} P_t^Q}{Y_{t_0} P_{t_0}^Y + Q_{t_0} P_{t_0}^Q} \quad (171)$$

where  $P_t^Y = am/Y_t$  and  $P_t^Q = (1-a)m/Q_t$  are the unit prices, in nominal dollars, of the non-housing consumption good and housing services, respectively.<sup>16</sup> This index systematically overstates inflation, by ignoring the fact that consumers can buy a less expensive, but equally desirable, basket of goods today, by substituting out of goods that have become relatively expensive into goods that have become relatively cheap.

The Paasche index measures today's cost of today's consumption basket relative to yesterday's cost of today's basket. That is, letting  $i^P(t, t_0)$  denote the time- $t$  Paasche index with base year  $t_0$ , which measures the time- $t$  price of the time- $t$  consumption basket relative to the price of the basket at time- $t_0$ ,

$$i^P(t, t_0) = \frac{Y_t P_t^Y + Q_t P_t^Q}{Y_t P_{t_0}^Y + Q_t P_{t_0}^Q}. \quad (172)$$

This index systematically understates inflation, by ignoring the fact that consumers could buy a less expensive, but equally desirable, basket of goods yesterday, by buying less of the goods that were relatively expensive yesterday and more of the goods that were relatively inexpensive.

Chained indices calculate changes in relative price levels over long horizons by multiplying together changes in relative price levels over intermediate "periods." For example, the change in the index from time- $t$  to time- $t+2$  is obtained by "chaining" (multiplying) together the change in the index from time- $t$  to time- $t+1$  and the change in the index from time- $t+1$  to time- $t+2$ . The  $n$ -period chained Laspeyres and Paasche indices

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<sup>16</sup> These prices hold provided the cash-in-advance constraint binds. The conditions that ensure that the constraint binds at all times follow the proof the proposition, in corollary C.1.



from  $t_0$  to  $t$  are then given by

$$i_n^L(t, t_0) \equiv \prod_{i=1}^n i^L \left( t_0 + \frac{i}{n}(t - t_0), t_0 + \frac{(i-1)}{n}(t - t_0) \right) \quad (173)$$

$$i_n^P(t, t_0) \equiv \prod_{i=1}^n i^P \left( t_0 + \frac{i}{n}(t - t_0), t_0 + \frac{(i-1)}{n}(t - t_0) \right). \quad (174)$$

The continuously chained Laspeyres and Paasche indices from  $t_0$  to  $t$  are defined by

$$i_t^{L,t_0} \equiv \lim_{n \rightarrow \infty} i_n^L(t, t_0) \quad (175)$$

$$i_t^{P,t_0} \equiv \lim_{n \rightarrow \infty} i_n^P(t, t_0), \quad (176)$$

and the continuously chained Fisher index is defined as

$$i_t^F \equiv \sqrt{i_t^{L,t_0} \cdot i_t^{P,t_0}}, \quad (177)$$

where for notational convenience we have suppressed dependence on the base year, because the choice of base year, which is irrelevant for relative price levels across time, is immaterial for our results.

The continuously chained Laspeyres and Paasche indices are multiplicative over sub-periods: given any  $t_1 \in (t_0, t)$ ,  $i_t^{L,t_0} = i_{t_1}^{L,t_0} i_t^{L,t_1}$ . Consequently, the change in one of these indices over an interval  $dt$ , scaled by its own level, is independent of the base year, so suppressing the base year in the notation we have

$$\begin{aligned} \frac{di_t^L}{i_t^L} &= \frac{Y_t P_{t+dt}^Y + Q_t P_{t+dt}^Q}{Y_t P_t^Y + Q_t P_t^Q} - 1 \\ &= a \left( \frac{dP_t^Y}{P_t^Y} \right) + (1-a) \left( \frac{dP_t^Q}{P_t^Q} \right) \\ &= -a \left( \frac{dY_t}{Y_t} \right) - (1-a) \left( \frac{dQ_t}{Q_t} \right) + a \left( \frac{dY_t}{Y_t} \right)^2 + (1-a) \left( \frac{dQ_t}{Q_t} \right)^2 \end{aligned} \quad (178)$$

and

$$\begin{aligned} \frac{di_t^P}{i_t^P} &= \frac{Y_{t+dt} P_{t+dt}^Y + Q_{t+dt} P_{t+dt}^Q}{Y_{t+dt} P_t^Y + Q_{t+dt} P_t^Q} - 1 \\ &= \frac{1}{1 + a \left( \frac{dP_t^Y}{P_t^Y} \right) + (1-a) \left( \frac{dP_t^Q}{P_t^Q} \right)} - 1 \\ &= -a \left( \frac{dY_t}{Y_t} \right) - (1-a) \left( \frac{dQ_t}{Q_t} \right) + \left( a \left( \frac{dY_t}{Y_t} \right) + (1-a) \left( \frac{dQ_t}{Q_t} \right) \right)^2. \end{aligned} \quad (179)$$

Then

$$\begin{aligned}
\frac{di_t^F}{i_t^F} &\equiv \frac{d\sqrt{i_t^L i_t^P}}{\sqrt{i_t^L i_t^P}} \\
&= -a \left( \frac{dY_t}{Y_t} \right) - (1-a) \left( \frac{dQ_t}{Q_t} \right) \\
&\quad + \frac{1}{2} \left( a \left( \frac{dY_t}{Y_t} \right)^2 + (1-a) \left( \frac{dQ_t}{Q_t} \right)^2 + \left( a \left( \frac{dY_t}{Y_t} \right) + (1-a) \left( \frac{dQ_t}{Q_t} \right) \right)^2 \right) \\
&= -a \left( \frac{dY_t}{Y_t} \right) - (1-a) \left( \frac{dQ_t}{Q_t} \right) + \frac{a(1-a)}{2} \left( \frac{dY_t}{Y_t} - \left( \frac{dQ_t}{Q_t} \right) \right)^2 \\
&\quad + \left( a \left( \frac{dY_t}{Y_t} \right) + (1-a) \left( \frac{dQ_t}{Q_t} \right) \right)^2 \\
&= -\frac{dC_t}{C_t} + \left( \frac{dC_t}{C_t} \right)^2 \\
&= \frac{dC_t^{-1}}{C_t^{-1}}.
\end{aligned} \tag{180}$$

Finally, note that a nominal dollar is a fixed ( $1/m$  fractional) claim to aggregate consumption, so nominal bonds *are* consumption bonds, and are priced accordingly. The payoff is, however, nominally risk-free, so the nominal interest rate is given by  $r_t^n(T) = -\ln \mathbf{E}_t \left[ e^{-\lambda T} C_{t,T}^{1-\gamma_C} \right] / T$ , so as a corollary to proposition 7.2

$$r_t^n(T) = \lambda + (\gamma_C - 1) \left( \mu_C - \gamma_C \frac{\sigma_C^2}{2} \right) - H \left( \frac{z_t^*}{z_t}, a(1-\gamma_C) \beta_{x,\xi}, b(1-\gamma_C), T \right). \tag{181}$$

The cash-in-advance constraint consequently binds at all times provided this rate is unambiguously positive. If this rate is always positive no agent will ever have an incentive to put a dollar under her mattress, because the dollar can always be loaned out at a positive rate. The parameter restrictions that guarantee the cash-in-advance constraint always binds are given in the following corollary.

**Corollary C.1.** *If  $\lambda + (\gamma_C - 1) \left( \mu_C - \gamma_C \frac{\sigma_C^2}{2} \right) > 0$  and  $\gamma_C \geq 1$  then the cash-in-advance constraint binds at all times.*

If the first condition holds, than nominal interest rates would be strictly positive absent the conversion technology. If the second condition holds, *i.e.*, if the representative consumer is more risk-averse than log-preferences, than the intertemporal smoothing motive dominates wealth effects and the conversion technology, which increases the expected growth rate of aggregate consumption, increases interest rates.

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